Annealed limit and quenched control for a diffusive disordered mean-field model with random jumps

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Point process : definitions

Point process (or counting process) Z :

- a random countable set of $\mathbb{R}_+$: $Z = \{T_i : i \in \mathbb{N}\}$
- a random point measure on $\mathbb{R}_+$: $Z = \sum_{i \in \mathbb{N}} \delta_{T_i}$

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A process λ is the **stochastic intensity** of Z if :

$$\forall 0 \leq a < b, \mathbb{E}\left[Z([a,b])|\mathcal{F}_a\right] = \mathbb{E}\left[\int_a^b \lambda_t dt \middle| \mathcal{F}_a\right]$$

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Characterization of P by A

$$P_t g(x) = g(x) + \int_0^t P_s A g(x) ds$$

Modeling in neuroscience

Neural network = directed graph

- **vertices** = neurons
- **arcs** = synapses

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$$\textbf{Neural activity} = \text{Set of spike times} = \text{Point process}$$

Spike rate depends on the potential of the neuron

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Network of N neurons :

- $$Z_t^{N,i} = \text{number of spikes of neuron } i \text{ emitted in } [0,t] = \text{point process with intensity } f(X_{t-}^{N,i})$$
- $X^{N,i}$ = potential of neuron i

$N-$neurons network model

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{N^\beta} \sum_{j=1}^{N} u_{ji}(t) dZ_t^{N,j}$$

with :

- $Z^{N,j}$ = **point process** with intensity $f(X_{t-}^{N,j})$
- $\beta = 1 \text{ or } 1/2$
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  $\rightarrow \alpha$ = leakage rate

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  $\rightarrow u^{ji}(t)/N^\beta =$ synaptic weight

Mean field limit

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- **linear scaling** $\beta = 1$ (LLN) :
  [Delattre et al. (2016)] (Hawkes process, $u_{ji}(t) = 1$), [Chevallier et al. (2019)] and [Agathe-Nerine (2022)] ($u_{ji}(t) = w(v_j, v_i)$ random)

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- **diffusive scaling** $\beta = 1/2$ (CLT)
  $\longrightarrow u_{ji}(t)$ random and centered

Diffusive scaling

- **Marked point processes**

- **Random environment**

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Property : semimartingale and Markov structure

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Marked model inconsistancy :
roles of synapses can change at every spike

Diffusive scaling, random environment, dimension 1

$N-$**particle system** U_j iid centered

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Reference : [Pfaffelhuber, Rotter, Stiefel (2022)] 2 differences :

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- $\mathcal{L}(U_1) = 1/2\delta_1 + 1/2\delta_{-1} \Rightarrow$ gentle distribution

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Our assumption : U_j iid centered and

$$\mathbb{E}\left[e^{\alpha|U_1|}\right] < \infty \text{ for some } \alpha > 0; \quad \sigma^2 := \mathbb{E}\left[U_1^2\right]$$

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Limit system $W \sim \mathcal{N}(0, \sigma^2)$

$$d\bar{X}_t = b(\bar{X}_t)dt + Wf(\bar{X}_t)dt + \sigma\sqrt{f(\bar{X}_t)}dB_t$$

Heuristics for the limit system

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CLT coupling from KMT result

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Let U_j $(j \geq 1)$ iid centered, for some $\alpha > 0$,

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Then there exist $W^{[N]}$ i.d.$\sim \mathcal{N}(0, \sigma^2)$ and K such that :

$$\left|\frac{1}{\sqrt{N}}\sum_{j=1}^{N} U_j - W^{[N]}\right| \leq K\frac{\ln N}{\sqrt{N}} \text{ and } \mathbb{E}\left[e^{\gamma K}\right] < \infty \text{ for some } \gamma > 0$$

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Proof. consequence of :

- KMT theorem [Komlós, Major, Tusnády (1976)]
- reasonning of [Ethier, Kurtz (2005)]

Main result : annealed convergence and quenched control

Theorem

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- For all $t > 0, g \in C_b^3(\mathbb{R})$,

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- For all $t > 0, g \in C_b^3(\mathbb{R})$, almost surely,

$$\left| \mathbb{E}_{\mathcal{E}}\left[g(X_t^N) \right] - \mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N) \right] \right| = \mathcal{O}\left(\frac{(\ln N)^{C_t}}{\sqrt{N}} \right)$$

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Remark : same convergence speed for FIDI distribution

Infinitesimal generator

$N-$particle system

$$dX_t^N = b(X_t^N)dt + N^{-1/2}\sum_{j=1}^{N} U_j \int_0^\infty \mathbb{1}_{\{z\le f(X_{t-}^N)\}} d\pi^j(t,z)$$

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$$A_{\mathcal{E}}^N g(x) = b(x)g'(x) + f(x)\sum_{j=1}^N \left[g\left(x + \frac{U_j}{\sqrt{N}}\right) - g(x)\right]$$

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Difference of generators

$$\left|A^N_{\mathcal{E}} g(x) - \bar{A}^N_{\mathcal{E}} g(x)\right| \leq f(x) \left(\left| \frac{1}{\sqrt{N}} \sum_{j=1}^{N} U_j - W^{[N]} \right| \cdot |g'(x)| + \frac{1}{2} \left| \frac{1}{N} \sum_{j=1}^{N} U_j^2 - \sigma^2 \right| \cdot |g''(x)| + \frac{1}{6N\sqrt{N}} \sum_{j=1}^{N} |U_j|^3 \cdot ||g'''||_\infty \right)$$

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$$\leq ||f||_\infty ||g||_{3,\infty} \epsilon_N(\mathcal{E})$$

Semigroup

$$\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x) = \int_0^t P^N_{\mathcal{E},t-s} \left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right) \bar{P}^N_{\mathcal{E},s} g(x) ds$$

Semigroup

$$\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x) = \int_0^t P^N_{\mathcal{E},t-s} \left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right) \bar{P}^N_{\mathcal{E},s} g(x) ds$$

Sketch of proof : $u(s) := P^N_{\mathcal{E},t-s}\bar{P}^N_{\mathcal{E},s}g(x)$

Semigroup

$$\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x) = \int_0^t P^N_{\mathcal{E},t-s}\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right)\bar{P}^N_{\mathcal{E},s} g(x) ds$$

Sketch of proof : $u(s) := P^N_{\mathcal{E},t-s}\bar{P}^N_{\mathcal{E},s} g(x)$

$$\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x) = u(t) - u(0)$$

Semigroup

$$\boxed{\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x) = \int_0^t P^N_{\mathcal{E},t-s}\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right)\bar{P}^N_{\mathcal{E},s} g(x) ds}$$

Sketch of proof : $u(s) := P^N_{\mathcal{E},t-s}\bar{P}^N_{\mathcal{E},s}g(x)$

$$\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x) = u(t) - u(0)$$
$$= \int_0^t u'(s) ds$$

Semigroup

$$\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x) = \int_0^t P^N_{\mathcal{E},t-s} \left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right) \bar{P}^N_{\mathcal{E},s} g(x) ds$$

Sketch of proof : $u(s) := P^N_{\mathcal{E},t-s}\bar{P}^N_{\mathcal{E},s}g(x)$

$$\begin{aligned}\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x) =& u(t) - u(0) \\ =& \int_0^t u'(s)ds \\ =& \int_0^t \left[-\frac{d}{du}\left(P^N_{\mathcal{E},u}\bar{P}^N_{\mathcal{E},s}g(x)\right)\Big|_{u=t-s} + \frac{d}{du}\left(P^N_{\mathcal{E},t-s}\bar{P}^N_{\mathcal{E},u}g(x)\right)\Big|_{u=s}\right] ds\end{aligned}$$

Semigroup

$$\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x) = \int_0^t P^N_{\mathcal{E},t-s} \left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right) \bar{P}^N_{\mathcal{E},s} g(x) ds$$

Sketch of proof : $u(s) := P^N_{\mathcal{E},t-s} \bar{P}^N_{\mathcal{E},s} g(x)$

$$\begin{aligned}
\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x) =& u(t) - u(0) \\
=& \int_0^t u'(s) ds \\
=& \int_0^t \left[-\frac{d}{du} \left(P^N_{\mathcal{E},u} \bar{P}^N_{\mathcal{E},s} g(x) \right)\Big|_{u=t-s} + \frac{d}{du} \left(P^N_{\mathcal{E},t-s} \bar{P}^N_{\mathcal{E},u} g(x) \right)\Big|_{u=s} \right] ds \\
=& \int_0^t \left[-P^N_{\mathcal{E},t-s} A^N_{\mathcal{E}} \bar{P}^N_{\mathcal{E},s} g(x) + P^N_{\mathcal{E},t-s} \bar{A}^N_{\mathcal{E}} \bar{P}^N_{\mathcal{E},s} g(x) \right] ds
\end{aligned}$$

Semigroup convergence

$$\left|\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x)\right| \leq \int_0^t \left|P^N_{\mathcal{E},t-s}\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right)\bar{P}^N_{\mathcal{E},s} g(x)\right| ds$$

Semigroup convergence

$$\left|\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x)\right| \leq \int_0^t \left|P^N_{\mathcal{E},t-s}\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right)\bar{P}^N_{\mathcal{E},s} g(x)\right| ds$$

$$\leq \int_0^t \mathbb{E}_{\mathcal{E},X_0^N=x}\left[\left|\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right)\bar{P}^N_{\mathcal{E},s} g(X^N_{t-s})\right|\right] ds$$

Semigroup convergence

$$\begin{aligned}
\left|\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x)\right| &\leq \int_0^t \left| P^N_{\mathcal{E},t-s}\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right) \bar{P}^N_{\mathcal{E},s} g(x)\right| ds \\
&\leq \int_0^t \mathbb{E}_{\mathcal{E},X^N_0=x}\left[\left|\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right) \bar{P}^N_{\mathcal{E},s} g(X^N_{t-s})\right|\right] ds \\
&\leq ||f||_\infty \left(\int_0^t \left|\left|\bar{P}^N_{\mathcal{E},s} g\right|\right|_{3,\infty} ds\right) \epsilon_N(\mathcal{E})
\end{aligned}$$

Semigroup convergence

$$\begin{aligned}\left|\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x)\right| &\leq \int_0^t \left| P^N_{\mathcal{E},t-s}\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right)\bar{P}^N_{\mathcal{E},s} g(x)\right| ds \\ &\leq \int_0^t \mathbb{E}_{\mathcal{E},X^N_0=x}\left[\left|\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right)\bar{P}^N_{\mathcal{E},s} g(X^N_{t-s})\right|\right] ds \\ &\leq ||f||_\infty \left(\int_0^t \left|\left|\bar{P}^N_{\mathcal{E},s} g\right|\right|_{3,\infty} ds\right)\epsilon_N(\mathcal{E})\end{aligned}$$

Control of $||\bar{P}^N_{\mathcal{E},s} g(x)||_{3,\infty}$

$$\bar{P}^N_{\mathcal{E},s} g(x) = \mathbb{E}_{\mathcal{E},\bar{X}^N_0=x}\left[g(\bar{X}^N_s)\right]$$

Semigroup convergence

$$\left|\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x)\right| \leq \int_0^t \left|P^N_{\mathcal{E},t-s}\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right)\bar{P}^N_{\mathcal{E},s} g(x)\right| ds$$

$$\leq \int_0^t \mathbb{E}_{\mathcal{E},X_0^N=x}\left[\left|\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right)\bar{P}^N_{\mathcal{E},s} g(X^N_{t-s})\right|\right] ds$$

$$\leq ||f||_\infty \left(\int_0^t \left|\left|\bar{P}^N_{\mathcal{E},s} g\right|\right|_{3,\infty} ds\right) \epsilon_N(\mathcal{E})$$

Control of $||\bar{P}^N_{\mathcal{E},s} g(x)||_{3,\infty}$

$$\bar{P}^N_{\mathcal{E},s} g(x) = \mathbb{E}_{\mathcal{E},\bar{X}_0^N=x}\left[g(\bar{X}^N_s)\right] = \mathbb{E}_{\mathcal{E}}\left[g(\bar{X}^N_s(x))\right]$$

Semigroup convergence

$$\left|\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x)\right| \leq \int_0^t \left|P^N_{\mathcal{E},t-s}\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right)\bar{P}^N_{\mathcal{E},s} g(x)\right| ds$$
$$\leq \int_0^t \mathbb{E}_{\mathcal{E},X^N_0=x}\left[\left|\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right)\bar{P}^N_{\mathcal{E},s} g(X^N_{t-s})\right|\right] ds$$
$$\leq ||f||_\infty \left(\int_0^t \left|\left|\bar{P}^N_{\mathcal{E},s} g\right|\right|_{3,\infty} ds\right)\epsilon_N(\mathcal{E})$$

Control of $||\bar{P}^N_{\mathcal{E},s} g(x)||_{3,\infty}$

$$\bar{P}^N_{\mathcal{E},s} g(x) = \mathbb{E}_{\mathcal{E},\bar{X}^N_0=x}\left[g(\bar{X}^N_s)\right] = \mathbb{E}_{\mathcal{E}}\left[g(\bar{X}^N_s(x))\right]$$
$$\partial_x \bar{P}^N_{\mathcal{E},s} g(x) = \mathbb{E}_{\mathcal{E}}\left[(\partial_x \bar{X}^N_s(x)) g'(\bar{X}^N_s(x))\right]$$

Semigroup convergence

$$\begin{aligned}
\left|\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x)\right| &\le \int_0^t \left|P^N_{\mathcal{E},t-s}\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right)\bar{P}^N_{\mathcal{E},s} g(x)\right| ds \\
&\le \int_0^t \mathbb{E}_{\mathcal{E},X_0^N=x}\left[\left|\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right)\bar{P}^N_{\mathcal{E},s} g(X^N_{t-s})\right|\right] ds \\
&\le ||f||_\infty \left(\int_0^t \left|\left|\bar{P}^N_{\mathcal{E},s} g\right|\right|_{3,\infty} ds\right)\epsilon_N(\mathcal{E})
\end{aligned}$$

Control of $||\bar{P}^N_{\mathcal{E},s} g(x)||_{3,\infty}$

$$\begin{aligned}
\bar{P}^N_{\mathcal{E},s} g(x) =& \mathbb{E}_{\mathcal{E},\bar{X}_0^N=x}\left[g(\bar{X}^N_s)\right] = \mathbb{E}_{\mathcal{E}}\left[g(\bar{X}^N_s(x))\right] \\
\partial_x \bar{P}^N_{\mathcal{E},s} g(x) =& \mathbb{E}_{\mathcal{E}}\left[(\partial_x \bar{X}^N_s(x)) g'(\bar{X}^N_s(x))\right] \\
|\partial_x \bar{P}^N_{\mathcal{E},s} g(x)| \le& ||g'||_\infty \sup_x \mathbb{E}_{\mathcal{E}}\left[|\partial_x \bar{X}^N_s(x)|\right]
\end{aligned}$$

Semigroup convergence

$$\begin{aligned}\left|\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x)\right| &\leq \int_0^t \left|P^N_{\mathcal{E},t-s}\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right)\bar{P}^N_{\mathcal{E},s} g(x)\right| ds \\ &\leq \int_0^t \mathbb{E}_{\mathcal{E},X^N_0=x}\left[\left|\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right)\bar{P}^N_{\mathcal{E},s} g(X^N_{t-s})\right|\right] ds \\ &\leq \|f\|_\infty \left(\int_0^t \left\|\bar{P}^N_{\mathcal{E},s} g\right\|_{3,\infty} ds\right)\epsilon_N(\mathcal{E})\end{aligned}$$

Control of $||\bar{P}^N_{\mathcal{E},s} g(x)||_{3,\infty}$

$$\begin{aligned}\bar{P}^N_{\mathcal{E},s} g(x) =& \mathbb{E}_{\mathcal{E},\bar{X}^N_0=x}\left[g(\bar{X}^N_s)\right] = \mathbb{E}_{\mathcal{E}}\left[g(\bar{X}^N_s(x))\right] \\ \partial_x \bar{P}^N_{\mathcal{E},s} g(x) =& \mathbb{E}_{\mathcal{E}}\left[(\partial_x \bar{X}^N_s(x)) g'(\bar{X}^N_s(x))\right] \\ |\partial_x \bar{P}^N_{\mathcal{E},s} g(x)| \leq& ||g'||_\infty \sup_x \mathbb{E}_{\mathcal{E}}\left[|\partial_x \bar{X}^N_s(x)|\right] \leq C_s ||g'||_\infty e^{C_s |W^{[N]}|}\end{aligned}$$

Semigroup convergence

$$
\begin{aligned}
\left|\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x)\right| &\leq \int_0^t \left| P^N_{\mathcal{E},t-s}\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right)\bar{P}^N_{\mathcal{E},s} g(x)\right| ds \\
&\leq \int_0^t \mathbb{E}_{\mathcal{E},X^N_0=x}\left[\left|\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right)\bar{P}^N_{\mathcal{E},s} g(X^N_{t-s})\right|\right] ds \\
&\leq ||f||_\infty \left(\int_0^t \left|\left|\bar{P}^N_{\mathcal{E},s} g\right|\right|_{3,\infty} ds\right)\epsilon_N(\mathcal{E}) \\
&\leq C_{g,t} e^{C_t|W^{[N]}|}\epsilon_N(\mathcal{E})
\end{aligned}
$$

Control of $||\bar{P}^N_{\mathcal{E},s} g(x)||_{3,\infty}$

$$
\begin{aligned}
\bar{P}^N_{\mathcal{E},s} g(x) =& \mathbb{E}_{\mathcal{E},\bar{X}^N_0=x}\left[g(\bar{X}^N_s)\right] = \mathbb{E}_{\mathcal{E}}\left[g(\bar{X}^N_s(x))\right] \\
\partial_x \bar{P}^N_{\mathcal{E},s} g(x) =& \mathbb{E}_{\mathcal{E}}\left[(\partial_x \bar{X}^N_s(x)) g'(\bar{X}^N_s(x))\right] \\
|\partial_x \bar{P}^N_{\mathcal{E},s} g(x)| \leq& ||g'||_\infty \sup_x \mathbb{E}_{\mathcal{E}}\left[|\partial_x \bar{X}^N_s(x)|\right] \leq C_s ||g'||_\infty e^{C_s|W^{[N]}|}
\end{aligned}
$$

Annealed convergence and quenched control

$$\left|\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x)\right| \leq C_{g,t} e^{C_t |W^{[N]}|} \epsilon_N(\mathcal{E})$$

Annealed convergence and quenched control

$$\left|\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x)\right| \leq C_{g,t} e^{C_t |W^{[N]}|} \epsilon_N(\mathcal{E})$$

Annealed convergence and quenched control

$$\left|\mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right]\right| \leq C_{g,t} e^{C_t |W^{[N]}|} \epsilon_N(\mathcal{E})$$

Annealed convergence and quenched control

$$\left| \mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N) \right] - \mathbb{E}_{\mathcal{E}}\left[g(X_t^N) \right] \right| \leq C_{g,t} e^{C_t |W^{[N]}|} \epsilon_N(\mathcal{E})$$

Annealed convergence

$$\boxed{\mathbb{E}\left[\left| \mathbb{E}_{\mathcal{E}}\left[g(X_t^N) \right] - \mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N) \right] \right| \right] \leq C_{g,t} \mathbb{E}\left[\epsilon_N(\mathcal{E})^2 \right]^{1/2}}$$

Annealed convergence and quenched control

$$\left|\mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right]\right| \leq C_{g,t} e^{C_t |W^{[N]}|} \epsilon_N(\mathcal{E})$$

Annealed convergence

$$\boxed{\mathbb{E}\left[\left|\mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right]\right|\right] \leq C_{g,t}\mathbb{E}\left[\epsilon_N(\mathcal{E})^2\right]^{1/2}}$$

with $\epsilon_N(\mathcal{E}) \leq C\left(\left|\frac{1}{\sqrt{N}}\sum_{j=1}^N U_j - W^{[N]}\right| + \left|\frac{1}{N}\sum_{j=1}^N U_j^2 - \sigma^2\right| + N^{-3/2}\sum_{j=1}^N |U_j|^3\right)$

Annealed convergence and quenched control

$$\left|\mathbb{E}_{\mathcal{E}}\left[g(\bar{X}^N_t)\right] - \mathbb{E}_{\mathcal{E}}\left[g(X^N_t)\right]\right| \leq C_{g,t}e^{C_t|W^{[N]}|}\epsilon_N(\mathcal{E})$$

Annealed convergence

$$\boxed{\mathbb{E}\left[\left|\mathbb{E}_{\mathcal{E}}\left[g(X^N_t)\right] - \mathbb{E}_{\mathcal{E}}\left[g(\bar{X}^N_t)\right]\right|\right] \leq C_{g,t}\mathbb{E}\left[\epsilon_N(\mathcal{E})^2\right]^{1/2}}$$

with $\epsilon_N(\mathcal{E}) \leq C\left(\left|\frac{1}{\sqrt{N}}\sum_{j=1}^N U_j - W^{[N]}\right| + \left|\frac{1}{N}\sum_{j=1}^N U_j^2 - \sigma^2\right| + N^{-3/2}\sum_{j=1}^N |U_j|^3\right)$

Annealed convergence and quenched control

$$\left|\mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right]-\mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right]\right|\leq C_{g,t}e^{C_t|W^{[N]}|}\epsilon_N(\mathcal{E})$$

Annealed convergence

$$\boxed{\mathbb{E}\left[\left|\mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right]-\mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right]\right|\right]\leq C_{g,t}\mathbb{E}\left[\epsilon_N(\mathcal{E})^2\right]^{1/2}}$$

with $\epsilon_N(\mathcal{E})\leq C\left(K\frac{\ln N}{\sqrt{N}}+\left|\frac{1}{N}\sum_{j=1}^N U_j^2-\sigma^2\right|+N^{-3/2}\sum_{j=1}^N|U_j|^3\right)$

Annealed convergence and quenched control

$$\left|\mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right]\right| \leq C_{g,t} e^{C_t |W^{[N]}|} \epsilon_N(\mathcal{E})$$

Annealed convergence

$$\boxed{\mathbb{E}\left[\left|\mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right]\right|\right] \leq C_{g,t}\mathbb{E}\left[\epsilon_N(\mathcal{E})^2\right]^{1/2}}$$

with $\epsilon_N(\mathcal{E}) \leq C\left(K\frac{\ln N}{\sqrt{N}} + \left|\frac{1}{N}\sum_{j=1}^N U_j^2 - \sigma^2\right| + N^{-3/2}\sum_{j=1}^N |U_j|^3\right)$

Annealed convergence and quenched control

$$\left|\mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right]\right| \leq C_{g,t} e^{C_t |W^{[N]}|} \epsilon_N(\mathcal{E})$$

Annealed convergence

$$\boxed{\mathbb{E}\left[\left|\mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right]\right|\right] \leq C_{g,t} \frac{\ln N}{\sqrt{N}}}$$

with $\epsilon_N(\mathcal{E}) \leq C \left(K \frac{\ln N}{\sqrt{N}} + \left| \frac{1}{N} \sum_{j=1}^N U_j^2 - \sigma^2 \right| + N^{-3/2} \sum_{j=1}^N |U_j|^3 \right)$

Annealed convergence and quenched control

$$\left|\mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right]\right| \leq C_{g,t} e^{C_t |W^{[N]}|} \epsilon_N(\mathcal{E})$$

Annealed convergence

$$\boxed{\mathbb{E}\left[\left|\mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right]\right|\right] \leq C_{g,t}\frac{\ln N}{\sqrt{N}}}$$

with $\epsilon_N(\mathcal{E}) \leq C\left(K\frac{\ln N}{\sqrt{N}} + \left|\frac{1}{N}\sum_{j=1}^N U_j^2 - \sigma^2\right| + N^{-3/2}\sum_{j=1}^N |U_j|^3\right)$

Quenched control

- $\epsilon_N(\mathcal{E}) = \mathcal{O}\left(\frac{\ln N}{\sqrt{N}}\right) + \mathcal{O}\left(\sqrt{\frac{\ln \ln N}{N}}\right) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$

Annealed convergence and quenched control

$$\left|\mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right]\right| \leq C_{g,t} e^{C_t |W^{[N]}|} \epsilon_N(\mathcal{E})$$

Annealed convergence

$$\boxed{\mathbb{E}\left[\left|\mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right]\right|\right] \leq C_{g,t} \frac{\ln N}{\sqrt{N}}}$$

with $\epsilon_N(\mathcal{E}) \leq C \left(K \frac{\ln N}{\sqrt{N}} + \left| \frac{1}{N} \sum_{j=1}^{N} U_j^2 - \sigma^2 \right| + N^{-3/2} \sum_{j=1}^{N} |U_j|^3 \right)$

Quenched control

- $\epsilon_N(\mathcal{E}) = \mathcal{O}\left(\frac{\ln N}{\sqrt{N}}\right) + \mathcal{O}\left(\sqrt{\frac{\ln \ln N}{N}}\right) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) = \mathcal{O}\left(\frac{\ln N}{\sqrt{N}}\right)$

Annealed convergence and quenched control

$$\left|\mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right]\right| \leq C_{g,t} e^{C_t|W^{[N]}|} \epsilon_N(\mathcal{E})$$

Annealed convergence

$$\boxed{\mathbb{E}\left[\left|\mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right]\right|\right] \leq C_{g,t}\frac{\ln N}{\sqrt{N}}}$$

with $\epsilon_N(\mathcal{E}) \leq C\left(K\frac{\ln N}{\sqrt{N}} + \left|\frac{1}{N}\sum_{j=1}^{N} U_j^2 - \sigma^2\right| + N^{-3/2}\sum_{j=1}^{N}|U_j|^3\right)$

Quenched control

- $\epsilon_N(\mathcal{E}) = \mathcal{O}\left(\frac{\ln N}{\sqrt{N}}\right) + \mathcal{O}\left(\sqrt{\frac{\ln \ln N}{N}}\right) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) = \mathcal{O}\left(\frac{\ln N}{\sqrt{N}}\right)$
- There exists β BM, $|W^{[N]}| = |\beta_N|/\sqrt{N} = \mathcal{O}(\sqrt{\ln \ln N})$

Annealed convergence and quenched control

$$\left|\mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right]\right| \leq C_{g,t} e^{C_t|W^{[N]}|}\epsilon_N(\mathcal{E})$$

Annealed convergence

$$\boxed{\mathbb{E}\left[\left|\mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right]\right|\right] \leq C_{g,t}\frac{\ln N}{\sqrt{N}}}$$

with $\epsilon_N(\mathcal{E}) \leq C\left(K\frac{\ln N}{\sqrt{N}} + \left|\frac{1}{N}\sum_{j=1}^N U_j^2 - \sigma^2\right| + N^{-3/2}\sum_{j=1}^N |U_j|^3\right)$

Quenched control

- $\epsilon_N(\mathcal{E}) = \mathcal{O}\left(\frac{\ln N}{\sqrt{N}}\right) + \mathcal{O}\left(\sqrt{\frac{\ln\ln N}{N}}\right) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) = \mathcal{O}\left(\frac{\ln N}{\sqrt{N}}\right)$
- There exists β BM, $|W^{[N]}| = |\beta_N|/\sqrt{N} = \mathcal{O}(\sqrt{\ln\ln N})$

$$\boxed{\left|\mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right]\right| \leq C_{g,t} e^{C_t|W^{[N]}|}\epsilon_N(\mathcal{E})}$$

Annealed convergence and quenched control

$$\left|\mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right]\right| \leq C_{g,t} e^{C_t |W^{[N]}|} \epsilon_N(\mathcal{E})$$

Annealed convergence

$$\boxed{\mathbb{E}\left[\left|\mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right]\right|\right] \leq C_{g,t} \frac{\ln N}{\sqrt{N}}}$$

with $\epsilon_N(\mathcal{E}) \leq C \left(K \frac{\ln N}{\sqrt{N}} + \left| \frac{1}{N} \sum_{j=1}^{N} U_j^2 - \sigma^2 \right| + N^{-3/2} \sum_{j=1}^{N} |U_j|^3 \right)$

Quenched control

- $\epsilon_N(\mathcal{E}) = \mathcal{O}\left(\frac{\ln N}{\sqrt{N}}\right) + \mathcal{O}\left(\sqrt{\frac{\ln \ln N}{N}}\right) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) = \mathcal{O}\left(\frac{\ln N}{\sqrt{N}}\right)$
- There exists β BM, $|W^{[N]}| = |\beta_N|/\sqrt{N} = \mathcal{O}(\sqrt{\ln \ln N})$

$$\boxed{\left|\mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right]\right| \leq C_{g,t} \mathcal{O}\left((\ln N)^{C_t} \cdot (\ln N)/\sqrt{N}\right)}$$

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Thank you for your attention !

Questions ?