Conditional propagation of chaos for mean field particle systems

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Joint work with: Eva Löcherbach [2] and Dasha Loukianova [3]

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Point process : definitions

A **point process** Z is:

- a random countable set of $\mathbb{R}_+$: $Z = \{T_i : i \in \mathbb{N}\}$
- a random point measure on $\mathbb{R}_+$: $Z = \sum_{i \in \mathbb{N}} \delta_{T_i}$

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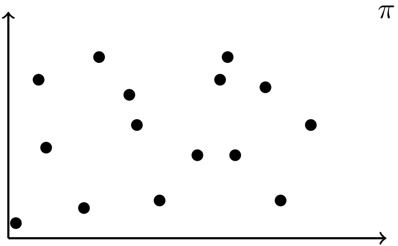
A process λ is the **stochastic intensity** of Z if:

$$\forall 0 \leq a < b, \mathbb{E}\left[Z([a,b])|\mathcal{F}_a\right] = \mathbb{E}\left[\int_a^b \lambda_t dt \middle| \mathcal{F}_a\right]$$

Thinning

π Poisson measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dt.dz$



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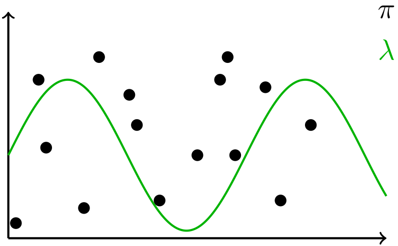

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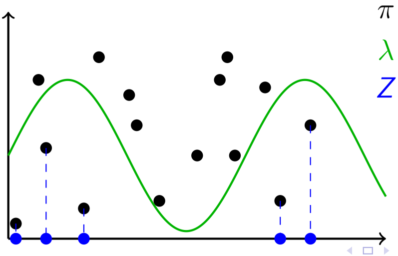
$$Z(A) = \int_{A \times \mathbb{R}_+} \mathbb{1}_{\{z \le \lambda(t)\}} d\pi(t,z)$$



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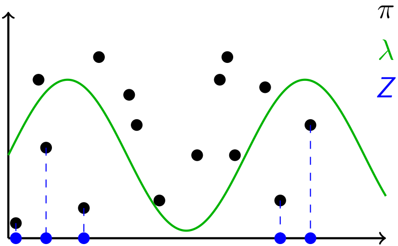
Then : λ is the stochastic intensity of Z



Interacting particle systems: Applications

$N-$particle system:

$$Z_t^{N,i} = \int_0^t \int_0^\infty \mathbb{1}_{\left\{z \le f(X_{s-}^{N,i})\right\}} d\pi^i(s,z)$$

$$dX_t^{N,i} = b(X_t^{N,i})dt + \sum_{j=1}^{N} \int_0^\infty u^{ji}(t) \mathbb{1}_{\left\{z \le f(X_{t-}^{N,j})\right\}} d\pi^j(t,z)$$

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- $Z_t^{N,i}$ process counts events of i
- each event excites/inhibits activities of all particles

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Applications:

- excitation/inhibition: book order, financial contagion,...
- mean field limit: mean field games

Mean field limit

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- linear scaling N^{-1} (LLN):
  [Delattre et al. (2016)] (Hawkes process, $u^{ji}(t) = 1$),
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- diffusive scaling $N^{-1/2}$ (CLT):
  [E. et al. (2022)] random and centered $u^{ji}(s)$

Diffusive scaling

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j\neq i}}^{N} \int_0^\infty \int_{\mathbb{R}} u \mathbb{1}_{\left\{z \le f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u) - \int_0^\infty \int_{\mathbb{R}} X_{t-}^{N,i} \mathbb{1}_{\left\{z \le f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u)$$

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- $X_t^{N,i} = X_s^{N,i} e^{-\alpha(t-s)}$ if the system does not jump in $[s,t]$

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- $X_t^{N,i} = 0$ if particle i creates event at t

Other models

Our N-particle system:

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j\neq i}}^{N} \int_0^\infty \int_{\mathbb{R}} u \mathbb{1}_{\left\{z \le f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u) - \int_0^\infty \int_{\mathbb{R}} X_{t-}^{N,i} \mathbb{1}_{\left\{z \le f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u)$$

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Another N-particle system:

$$dX_t^{N,i} = b(X_t^{N,i}) dt + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_0^\infty \int_{\mathbb{R}} u \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u) + \sigma(X_t^{N,i}) dW_t^i$$

with W^i independent BM

Limit system: heuristic (1)

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}} u \mathbb{1}_{\left\{ z \leq f(X_{t-}^{N,j}) \right\}} d\pi^j(t,z,u)$$
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$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + d\bar{M}_t$$

$$- \bar{X}_{t-}^i \int_{\mathbb{R}_+\times\mathbb{R}} \mathbb{1}_{\left\{z \le f(\bar{X}_{t-}^i)\right\}} d\pi^i(t,z,u)$$

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$$\langle \bar{M} \rangle_t = \lim_N \langle M^N \rangle_t = \lim_N \sigma^2 \int_0^t \frac{1}{N} \sum_{j=1}^{N} f(X_s^{N,j}) ds$$

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Then $\bar{M}$ should satisfy

$$\bar{M}_t = \sigma \int_0^t \sqrt{\lim_N \frac{1}{N} \sum_{j=1}^{N} f(\bar{X}_s^j)} dW_s = \sigma \int_0^t \sqrt{\lim_N \bar{\mu}_s^N(f)} dW_s$$

with $\bar{\mu}^N := \frac{1}{N} \sum_{j=1}^{N} \delta_{\bar{X}^j}$

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Well-posedness of the limit equation

$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mathbb{E}\left[f(\bar{X}_t^i)|W\right]} dW_t - \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\{z \leq f(\bar{X}_{t-}^i)\}} d\pi^i(t,z,u)$$

Problems:

- conditional expectation in the Brownian term
- unbounded jumps
- jump term and Brownian term

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Assumptions:

- $0 < \inf f$ and $\sup f < \infty$
- $x \mapsto f(x)$ and $x \mapsto xf(x)$ Lipschitz

Result: the system $(\bar{X}^i)_{i \geq 1}$ is strongly well-posed

Example of function f

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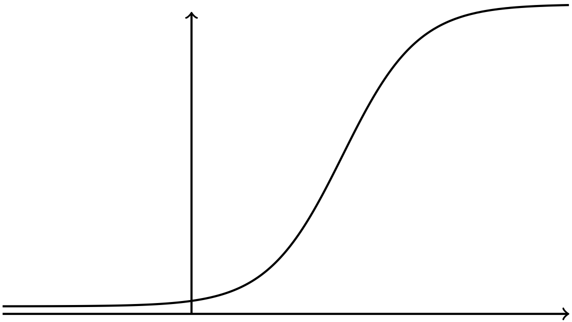

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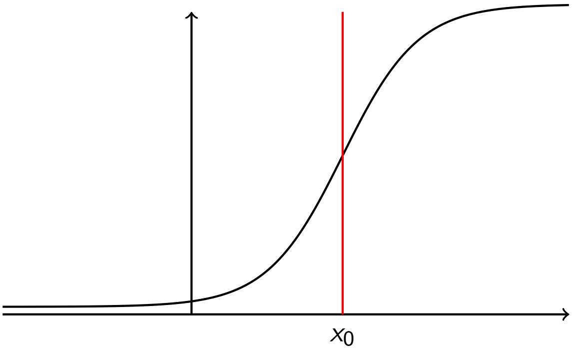

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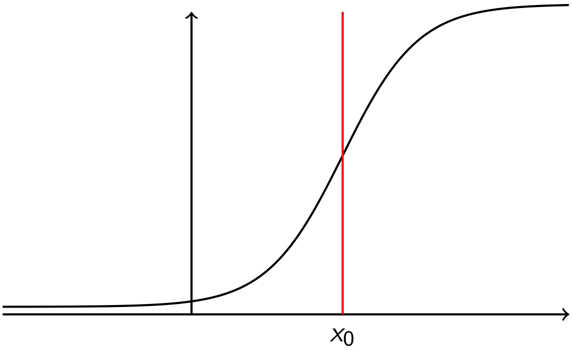
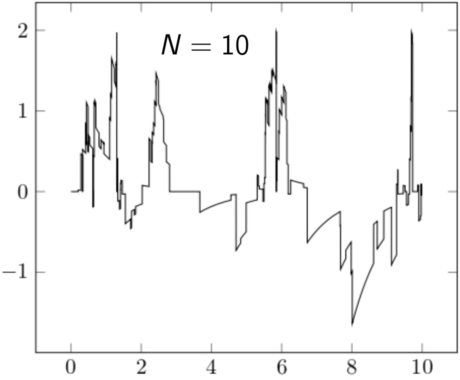
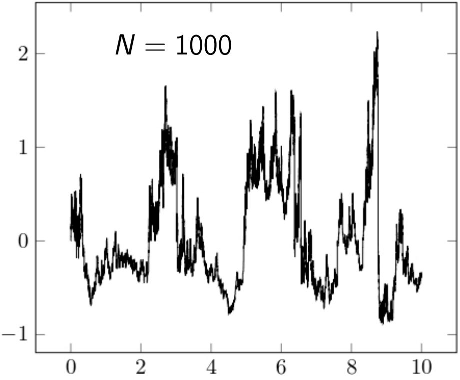


"Particle i active / inactive" $\approx$ "$X^{N,i} > x_0$ / $X^{N,i} < x_0$"

Simulations of $X^{N,1}$





Another version of the limit system

Strong limit system:

$$d\bar{X}^i_t = -\alpha \bar{X}^i_t dt + \sigma \sqrt{\mathbb{E}\left[f(\bar{X}^i_t) | W \right]} dW_t$$
$$- \bar{X}^i_{t-} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\left\{ z \leq f(\bar{X}^i_{t-}) \right\}} d\pi^i(t,z,u)$$

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Weak limit system:

$$d\bar{Y}_t^i = -\alpha \bar{Y}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t - \bar{Y}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\{z \leq f(\bar{Y}_{t-}^i)\}} d\pi^i(t,z,u)$$

where $\mu_t = \mathcal{L}(\bar{Y}_t^1 | \mu_t)$ is the **directing measure** of $(\bar{Y}_t^i)_{i \geq 1}$

Equivalence between the two systems

Auxiliary system:

$$d\tilde{X}_t^{N,i} = -\alpha \tilde{X}_t^{N,i} dt + \sigma \sqrt{\frac{1}{N}\sum_{j=1}^{N} f(\tilde{X}_t^{N,j})} dW_t - \tilde{X}_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\left\{z \leq f(\tilde{X}_{t-}^{N,i})\right\}} d\pi^i(t,z,u)$$

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For $0 \leq t \leq T$ (small enough)

$$u_N(t) \leq CN^{-1/2} \underset{N\to\infty}{\longrightarrow} 0$$

Convergence of $(X^{N,i})_{1\leq i\leq N}$

$$dX_t^{N,i} = -\alpha X_t^{N,i}dt + \frac{1}{\sqrt{N}}\sum_{j\neq i}\int_{\mathbb{R}_+\times\mathbb{R}} u\mathbb{1}_{\left\{z\leq f(X_{t-}^{N,j})\right\}}d\pi^j(t,z,u)$$
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Result [E., Löcherbach, Loukianova (2021)]

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i.e. $\mu^N := \sum_{j=1}^N \delta_{X^{N,j}}$ converges to $\mu := \mathcal{L}(\bar{X}^1|W)$ in $\mathcal{P}(D)$

Outline of the proof

Step 1. $(\mu^N)_N$ is tight on $\mathcal{P}(D)$
Equivalent condition: $(X^{N,1})_N$ is tight on D
Proof: Aldous' criterion

Step 2. Identifying the limit distribution of $(\mu^N)_N$
Proof: any limit of μ^N is solution of a martingale problem

Martingale problem

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$$Lg(m, x^1, x^2) = -\alpha x^1 \partial_1 g(x) - \alpha x^2 \partial_2 g(x) + \frac{\sigma^2}{2} m(f) \sum_{i,j=1}^{2} \partial^2_{i,j} g(x)$$
$$+ f(x^1)(g(0, x^2) - g(x)) + f(x^2)(g(x^1, 0) - g(x))$$

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Then the law of $\mu = \mathcal{L}(\bar{Y}^1 | W)$ is uniquely determined

Convergence of μ^N to the solution of $(\mathcal{M})$

Let μ be the limit of (a subsequence of) μ^N
$\mathcal{L}(\mu)$ is solution of $(\mathcal{M})$ if

$$\mathbb{E}\left[F(\mu)\right] = 0$$

for any F of the form

$$F(m) := \int_{D^2} m \otimes m(d\gamma)\phi_1(\gamma_{s_1})...\phi_k(\gamma_{s_k})\Big[\phi(\gamma_t) - \phi(\gamma_s) - \int_s^t L\phi(m_r, \gamma_r)dr\Big]$$

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The expression of $F(\mu^N)$

$$F(\mu^N) :=$$

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\phi_1(X^{N,i}_{s_1},X^{N,j}_{s_1})\ldots\phi_k(X^{N,i}_{s_k},X^{N,j}_{s_k})\Big[\phi(X^{N,i}_t,X^{N,j}_t)-\phi(X^{N,i}_s,X^{N,j}_s)$$

$$+\alpha\int_s^t X^{N,i}_r\partial_1\phi(X^{N,i}_r,X^{N,j}_r)dr+\alpha\int_s^t X^{N,j}_r\partial_2\phi(X^{N,i}_r,X^{N,j}_r)dr$$

$$-\int_s^t f(X^{N,i}_r)(\phi(0,X^{N,j}_r)-\phi(X^{N,i}_r,X^{N,j}_r))dr$$

$$-\int_s^t f(X^{N,j}_r)(\phi(X^{N,i}_r,0)-\phi(X^{N,i}_r,X^{N,j}_r))dr$$

$$-\int_s^t\int_{\mathbb{R}}\sum_{k=1}^{N} f(X^{N,k}_r)\frac{u^2}{2N}\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X^{N,i}_r,X^{N,j}_r)d\nu(u)dr\Big]$$

The expression of $\phi(X^{N,i}, X^{N,j})$

By Ito's formula,

$$\mathbb{E}\phi(X_t^{N,i}, X_t^{N,j}) - \phi(X_s^{N,i}, X_s^{N,j}) =$$

$$\mathbb{E} - \alpha \int_s^t X_r^{N,i} \partial_1 \phi(X_r^{N,i}, X_r^{N,j}) dr - \alpha \int_s^t X_r^{N,j} \partial_2 \phi(X_r^{N,i}, X_r^{N,j}) dr$$

$$+ \int_s^t \int_{\mathbb{R}} f(X_r^{N,i}) (\phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j})) d\nu(u) dr$$

$$+ \int_s^t \int_{\mathbb{R}} f(X_r^{N,j}) (\phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, 0) - \phi(X_r^{N,i}, X_r^{N,j})) d\nu(u) dr$$

$$+ \int_s^t \int_{\mathbb{R}} \sum_{\substack{k=1 \\ k \neq i,j}}^{N} f(X_r^{N,k}) (\phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j})) d\nu(u) dr$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

The reset jump term

$$\left|\phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}})\right|$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

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$$\left|\phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}})\right| \leq C\frac{|u|}{\sqrt{N}}$$

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The small jump term

$$N\left|\phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j}) - \frac{u^2}{2N}\sum_{i_1,i_2=1}^{2} \partial^2_{i_1,i_2}\phi(X_r^{N,i}, X_r^{N,j})\right|$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

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$$\left. - \frac{u}{\sqrt{N}}\sum_{i_1=1}^{2}\partial_{i_1}\phi(X_r^{N,i}, X_r^{N,j}) - \frac{u^2}{2N}\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i}, X_r^{N,j})\right|$$

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$$\leq CN\frac{|u|^3}{N\sqrt{N}} = CN^{-1/2}|u|^3$$

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$$\leq CN\frac{|u|^3}{N\sqrt{N}} = CN^{-1/2}|u|^3$$

$$CN^{-1/2} \geq \mathbb{E}\left[F(\mu^N)\right] \underset{N\to\infty}{\longrightarrow} \mathbb{E}\left[F(\mu)\right] = 0$$

Convergence of $(\mu^N)_N$

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j \neq i} \int_{\mathbb{R}_+ \times \mathbb{R}} u \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u)$$
$$- X_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u)$$

$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mathbb{E}[f(\bar{X}_t^i) | \sigma(W)]} dW_t$$
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- $(\mu^N)_N$ is tight on $\mathcal{P}(D)$

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- $(\mu^N)_N$ is tight on $\mathcal{P}(D)$
- let μ be the limit of a converging subsequence
- $\mathcal{L}(\mu)$ is the unique solution of $(\mathcal{M})$
- $\mu = \mathcal{L}(\bar{X}^1|W)$ is the only limit of $(\mu^N)_N$

McKean-Vlasov model

$$dX_t^{N,i} = b(X_t^{N,i}, \mu_t^N)dt + \sigma(X_t^{N,i}, \mu_t^N)d\beta_t^i$$

$$+ \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}^{\mathbb{N}^*}} \Psi(X_{t-}^{N,k}, X_{t-}^{N,i}, \mu_{t-}^N, u^k, u^i) \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,k}, \mu_{t-}^N)\right\}} d\pi^k(t,z,u)$$

with $\mu^N := \frac{1}{N} \sum_{i=1}^N \delta_{X^{N,i}}$, and π^k has intensity $dt \cdot dz \cdot \nu(du)$ $(\nu = \nu_1^{\otimes \mathbb{N}^*})$

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Dynamic of $X^{N,i}$:

- while there is no jump, the dynamic is given by the drift and Brownian terms
- if there is a jump at time t, created by particle k, each particle i creates a r.v. U^i (the U^i are i.i.d.),

$$X_t^{N,i} = X_{t-}^{N,i} + \frac{1}{\sqrt{N}} \Psi(X_{t-}^{N,k}, X_{t-}^{N,i}, \mu_{t-}^N, U^k, U^i)$$

Heuristics for the limit system

$$dX_t^{N,i} = b(X_t^{N,i}, \mu_t^N)dt + \sigma(X_t^{N,i}, \mu_t^N)d\beta_t^i$$

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$$\langle J^{N,i}, J^{N,j} \rangle_t = \frac{1}{N} \sum_{k=1}^{N} \int_0^t \int_{\mathbb{R}^{\mathbb{N}^*}}$$

$$\Psi(X_s^{N,k}, X_s^{N,i}, \mu_s^N, u^k, u^i) \Psi(X_s^{N,k}, X_s^{N,j}, \mu_s^N, u^k, u^j) f(X_s^{N,k}, \mu_s^N) \nu(du) ds$$

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Particular case

$$\Psi(x, y, m, u, v) = \Psi(u, v)$$

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$$= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}} \Psi(u^1, u^i)^2 f(x, \mu_s^N)\nu(du)\mu_s^N(dx)ds \quad \text{if } i = j$$

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$$= \varsigma^2 \int_0^t \int_{\mathbb{R}} f(x, \mu_s^N)\mu_s^N(dx)ds \quad \text{if } i = j$$

with $\varsigma^2 = \int \Psi(u^1, u^2)^2 \nu(du)$

Particular case

$$\Psi(x, y, m, u, v) = \Psi(u, v)$$

$$\begin{aligned}
\langle J^{N,i}, J^{N,j} \rangle_t = & \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}} \Psi(u^1, u^i)\Psi(u^1, u^j) f(x, \mu_s^N)\nu(du)\mu_s^N(dx)ds \\
= & \varsigma^2 \int_0^t \int_{\mathbb{R}} f(x, \mu_s^N)\mu_s^N(dx)ds \quad \text{if } i = j \\
= & \kappa^2 \int_0^t \int_{\mathbb{R}} f(x, \mu_s^N)\mu_s^N(dx)ds \quad \text{if } i \neq j
\end{aligned}$$

with $\varsigma^2 = \int \Psi(u^1, u^2)^2 \nu(du)$ and $\kappa^2 = \int \Psi(u^1, u^2)\Psi(u^1, u^3)\nu(du)$

Particular case

$$\Psi(x, y, m, u, v) = \Psi(u, v)$$

$$\begin{aligned}\langle J^{N,i}, J^{N,j}\rangle_t &= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}} \Psi(u^1, u^i)\Psi(u^1, u^j) f(x, \mu_s^N)\nu(du)\mu_s^N(dx)ds \\ &= \varsigma^2 \int_0^t \int_{\mathbb{R}} f(x, \mu_s^N)\mu_s^N(dx)ds \quad \text{if } i = j \\ &= \kappa^2 \int_0^t \int_{\mathbb{R}} f(x, \mu_s^N)\mu_s^N(dx)ds \quad \text{if } i \neq j\end{aligned}$$

with $\varsigma^2 = \int \Psi(u^1, u^2)^2 \nu(du)$ and $\kappa^2 = \int \Psi(u^1, u^2)\Psi(u^1, u^3)\nu(du)$

$$\bar{J}_t^i = \kappa \int_0^t \sqrt{\int_{\mathbb{R}} f(x, \mu_s)\mu_s(dx)} dW_s + \sqrt{\varsigma^2 - \kappa^2} \int_0^t \sqrt{\int_{\mathbb{R}} f(x, \mu_s)\mu_s(dx)} dW_s^i$$

with W, W^i i.i.d. Brownian motions and $\mu = \mathcal{L}(\bar{X}^i | W)$

General case

$$\langle J^{N,i}, J^{N,j} \rangle_t = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}}$$

$$\Psi(x, X_s^{N,i}, \mu_s^N, u^1, u^i) \Psi(x, X^{N,j}, \mu_s^N, u^1, u^j) f(x, \mu_s^N) \nu(du) \mu_s^N(dx) ds$$

General case

$$\langle J^{N,i}, J^{N,j} \rangle_t = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}}$$
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Problem: blue term is not a product, but integral of a product

General case

$$\langle J^{N,i}, J^{N,j}\rangle_t = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}} \Psi(x, X_s^{N,i}, \mu_s^N, u^1, u^i)\Psi(x, X^{N,j}, \mu_s^N, u^1, u^j) f(x, \mu_s^N)\nu(du)\mu_s^N(dx) ds$$

Problem: blue term is not a product, but integral of a product

Solution: let $M(dt, dz) = M_t(dz)$ be a martingale measure on $\mathbb{R}_+ \times E$ with intensity $dt \cdot m_t(dz)$,

$$\langle M_.(A), M_.(B)\rangle_t = \int_0^t \int_E \mathbb{1}_A(z) \cdot \mathbb{1}_B(z) m_s(dz) ds$$

General case

$$\langle J^{N,i}, J^{N,j}\rangle_t = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}}$$
$$\Psi(x, X_s^{N,i}, \mu_s^N, u^1, u^i)\Psi(x, X^{N,j}, \mu_s^N, u^1, u^j) f(x, \mu_s^N)\nu(du)\mu_s^N(dx)ds$$

Problem: blue term is not a product, but integral of a product

Solution: let $M(dt, dz) = M_t(dz)$ be a martingale measure on $\mathbb{R}_+ \times E$ with intensity $dt \cdot m_t(dz)$,

$$\langle M.(A), M.(B)\rangle_t = \int_0^t \int_E \mathbb{1}_A(z) \cdot \mathbb{1}_B(z) m_s(dz)ds$$

Here: $E = \mathbb{R}^{\mathbb{N}^*} \times \mathbb{R}$ and $m_s(du, dx) = \nu(du) \cdot \mu_s(dx)$

Limit system (1)

$$
\begin{aligned}
d\bar{X}^i_t =& b(\bar{X}^i_t,\mu_t)dt + \sigma(\bar{X}^i_t,\mu_t)d\beta^i_t \\
&+ \int \tilde{\Psi}(x,\bar{X}^i_t,\mu_t,v)\sqrt{f(x,\mu_t)}dM(t,x,v) \\
&+ \int \kappa(x,\bar{X}^i_t,\mu_t)\sqrt{f(x,\mu_t)}dM^i(t,x)
\end{aligned}
$$

with:

- M martingale measure intensity $dt\mu_t(dx)\nu_1(dv)$
- M^i martingale measure intensity $dt\mu_t(dx)$

$$
\tilde{\Psi}(x,y,m,v) = \int_{\mathbb{R}^{\mathbb{N}^*}} \Psi(x,y,m,v,u^1)\nu(du)
$$

$$
\kappa(x,y,m)^2 = \int \Psi(x,y,m,u^1,u^2)^2\nu(du) - \int \tilde{\Psi}(x,y,m,u^1)^2\nu(du)
$$

Limit system (2)

$$
\begin{aligned}
d\bar{X}_t^i = & b(\bar{X}_t^i, \mu_t)dt + \sigma(\bar{X}_t^i, \mu_t)d\beta_t^i \\
& + \int \tilde{\Psi}(x, \bar{X}_t^i, \mu_t, v)\sqrt{f(x, \mu_t)}dM(t, x, v) \\
& + \int \kappa(x, \bar{X}_t^i, \mu_t)\sqrt{f(x, \mu_t)}dM^i(t, x)
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Limit system (2)

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\end{aligned}
$$

M and M^i are orthogonal (not independent):

$$
M_t^i(A) = \int_0^t \int_0^1 \mathbb{1}_A(F_s^{-1}(p))dW^i(s, p)
$$

$$
M_t(A \times B) = \int_0^t \int_0^1 \int_{\mathbb{R}} \mathbb{1}_A(F_s^{-1}(p))\mathbb{1}_B(u)dW(s, p, u)
$$

Limit system (2)

$$d\bar{X}_t^i = b(\bar{X}_t^i, \mu_t)dt + \sigma(\bar{X}_t^i, \mu_t)d\beta_t^i + \int \tilde{\Psi}(x, \bar{X}_t^i, \mu_t, v)\sqrt{f(x, \mu_t)}dM(t, x, v) + \int \kappa(x, \bar{X}_t^i, \mu_t)\sqrt{f(x, \mu_t)}dM^i(t, x)$$

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with:

- W^i, W independent WN with intensities $dtdp$ and $dtdp\nu_1(du)$

Limit system (2)

$$d\bar{X}^i_t = b(\bar{X}^i_t, \mu_t)dt + \sigma(\bar{X}^i_t, \mu_t)d\beta^i_t + \int \tilde{\Psi}(x, \bar{X}^i_t, \mu_t, v)\sqrt{f(x, \mu_t)}dM(t, x, v) + \int \kappa(x, \bar{X}^i_t, \mu_t)\sqrt{f(x, \mu_t)}dM^i(t, x)$$

M and M^i are orthogonal (not independent):

$$M^i_t(A) = \int_0^t \int_0^1 \mathbb{1}_A(F_s^{-1}(p))dW^i(s, p)$$

$$M_t(A \times B) = \int_0^t \int_0^1 \int_{\mathbb{R}} \mathbb{1}_A(F_s^{-1}(p))\mathbb{1}_B(u)dW(s, p, u)$$

with:

- W^i, W independent WN with intensities $dtdp$ and $dtdp\nu_1(du)$
- F_s^{-1} = generalized inverse distribution function of μ_s

Limit system (2)

$$d\bar{X}^i_t = b(\bar{X}^i_t, \mu_t)dt + \sigma(\bar{X}^i_t, \mu_t)d\beta^i_t + \int \tilde{\Psi}(x, \bar{X}^i_t, \mu_t, v)\sqrt{f(x, \mu_t)}dM(t, x, v) + \int \kappa(x, \bar{X}^i_t, \mu_t)\sqrt{f(x, \mu_t)}dM^i(t, x)$$

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with:

- W^i, W independent WN with intensities $dtdp$ and $dtdp\nu_1(du)$
- F_s^{-1} = generalized inverse distribution function of μ_s
- $\mu_s = \mathcal{L}(\bar{X}^i_s | W)$

Generator of the limit system

$$
\begin{aligned}
d\bar{X}_t^i =& b(\bar{X}_t^i, \mu_t)dt + \sigma(\bar{X}_t^i, \mu_t)d\beta_t^i \\
&+ \int \tilde{\Psi}(x, \bar{X}_t^i, \mu_t, v)\sqrt{f(x, \mu_t)}dM(t, x, v) \\
&+ \int \kappa(x, \bar{X}_t^i, \mu_t)\sqrt{f(x, \mu_t)}dM^i(t, x)
\end{aligned}
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Generator of the limit system

$$
\begin{aligned}
d\bar{X}^i_t = & b(\bar{X}^i_t, \mu_t)dt + \sigma(\bar{X}^i_t, \mu_t)d\beta^i_t \\
& + \int \tilde{\Psi}(x, \bar{X}^i_t, \mu_t, v)\sqrt{f(x, \mu_t)}dM(t, x, v) \\
& + \int \kappa(x, \bar{X}^i_t, \mu_t)\sqrt{f(x, \mu_t)}dM^i(t, x)
\end{aligned}
$$

$$
\begin{aligned}
& Lg(y, m, x, v) = b(y^1, m)\partial_{y^1} g(y) + b(y^2, m)\partial_{y^2} g(y) \\
& + \frac{1}{2}\sigma(y^1, m)^2 \partial^2_{y^1} g(y) + \frac{1}{2}\sigma(y^2, m)^2 \partial^2_{y^2} g(y) \\
& + \frac{1}{2} f(x, m)\kappa(x, y^1, m)^2 \partial^2_{y^1} g(y) + \frac{1}{2} f(x, m)\kappa(x, y^2, m)^2 \partial^2_{y^2} g(y) \\
& + \frac{1}{2} f(x, m) \sum_{i,j=1}^{2} \tilde{\Psi}(x, y^i, m, v)\tilde{\Psi}(x, y^j, m, v)\partial^2_{y^i y^j} g(y)
\end{aligned}
$$

Generator of the limit system

$$
\begin{aligned}
d\bar{X}^i_t =& b(\bar{X}^i_t, \mu_t)dt + \sigma(\bar{X}^i_t, \mu_t)d\beta^i_t \\
&+ \int \tilde{\Psi}(x, \bar{X}^i_t, \mu_t, v)\sqrt{f(x,\mu_t)}dM(t,x,v) \\
&+ \int \kappa(x, \bar{X}^i_t, \mu_t)\sqrt{f(x,\mu_t)}dM^i(t,x)
\end{aligned}
$$

$$
\begin{aligned}
Lg(y,m,x,v) =& \; b(y^1,m)\partial_{y^1}g(y) + b(y^2,m)\partial_{y^2}g(y) \\
&+ \frac{1}{2}\sigma(y^1,m)^2\partial^2_{y^1}g(y) + \frac{1}{2}\sigma(y^2,m)^2\partial^2_{y^2}g(y) \\
&+ \frac{1}{2}f(x,m)\kappa(x,y^1,m)^2\partial^2_{y^1}g(y) + \frac{1}{2}f(x,m)\kappa(x,y^2,m)^2\partial^2_{y^2}g(y) \\
&+ \frac{1}{2}f(x,m)\sum_{i,j=1}^{2}\tilde{\Psi}(x,y^i,m,v)\tilde{\Psi}(x,y^j,m,v)\partial^2_{y^iy^j}g(y)
\end{aligned}
$$

Representation theorems: [El Karoui & Méléard (1990)]

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Thank you for your attention !

Questions ?