Annealed and quenched limits for a diffusive disordered mean-field model with random jumps

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1. Mathematical background
   - Point processes
   - Thinning
2. Model
   - Neural networks model
   - Definitions of the systems
   - Heuristics
3. Convergence
   - Result
   - Finite-dimensional convergence
   - Tightness
4. Prospect ?

Point process : definitions

Point process (or counting process) Z :

- a random countable set of $\mathbb{R}_+$: $Z = \{T_i : i \in \mathbb{N}\}$
- a random point measure on $\mathbb{R}_+$: $Z = \sum_{i \in \mathbb{N}} \delta_{T_i}$

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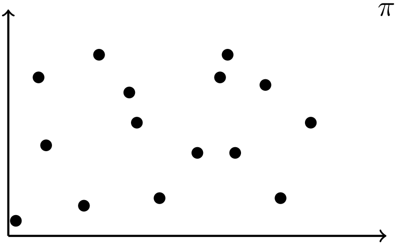
A process λ is the **stochastic intensity** of Z if :

$$\forall 0 \leq a < b, \mathbb{E}\left[Z([a,b]) | \mathcal{F}_a\right] = \mathbb{E}\left[\int_a^b \lambda_t dt \middle| \mathcal{F}_a\right]$$

Thinning

π Poisson measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dt.dz$



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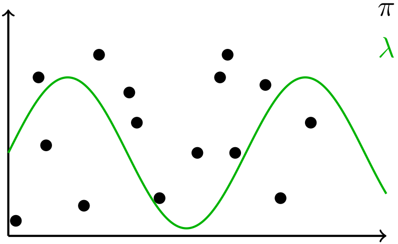

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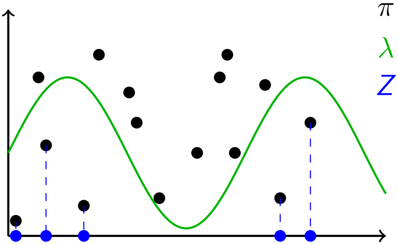

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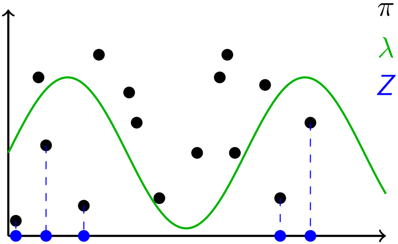

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Here, $X^{N,i}$ solves an SDE directed by $(Z^{N,j})_{1\leq j\leq N}$

$N-$neurons network model

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{N^\beta} \sum_{j=1}^{N} \int u^{ji}(t) \mathbb{1}_{\left\{z \le f(X_{t-}^{N,j})\right\}} d\pi^j(t,z)$$

with :

- π^j = PRM with intensity $dt \cdot dz$
- $\beta = 1 \text{ or } 1/2$
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- $X_t^{N,i} = X_{t-}^{N,i} + \frac{u^{ji}(t)}{N^\beta}$ if a neuron j emits a spike at t
  $\rightarrow u^{ji}(t)/N^\beta$ = synaptic weight

Mean field limit

$$dX_t^{N,i} = b(X_t^{N,i})dt + \frac{1}{N^\beta}\sum_{j=1}^N \int_0^\infty u^{ji}(t)\mathbb{1}_{\left\{z \le f(X_{t-}^{N,j})\right\}} d\pi^j(t,z)$$

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- linear scaling $\beta = 1$ (LLN) :
  [Delattre et al. (2016)] (Hawkes process, $u^{ji}(t) = 1$),
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- diffusive scaling $\beta = 1/2$ (CLT)

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  - scaling $N^{-1} \Longrightarrow$ limit ODE
  - scaling $N^{-1/2} \Longrightarrow$ limit SDE (model with noise)

$\Rightarrow$ various noises (the other neurons, the ion channels,...)

Diffusive scaling

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- **Random environment**

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Property : semimartingale and Markov structure

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Marked model inconsistancy :
roles of synapses can change at every spike

Diffusive scaling, random environment, dimension 1

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Limit system $W \sim \mathcal{N}(0,\sigma^2)$

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Heuristics for the limit system

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CLT coupling from KMT result

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Then there exist $W^{[N]}$ i.d.$\sim \mathcal{N}(0, \sigma^2)$ and K such that :

$$\left|\frac{1}{\sqrt{N}}\sum_{j=1}^{N} U_j - W^{[N]}\right| \leq K\frac{\ln N}{\sqrt{N}} \text{ and } \mathbb{E}\left[e^{\gamma K}\right] < \infty \text{ for some } \gamma > 0$$

Sketch of proof of CLT coupling

[Komlós, Major, Tusnády (1976)] :
there exist BM β and constants Γ, Λ, λ, such that $\forall N \geq 2, x > 0$,

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with $W^{[N]} := \sigma \beta_N / \sqrt{N} \sim \mathcal{N}(0, \sigma^2)$

Conditional stochastic calculus

$$dX_t^N = b(X_t^N)dt + N^{-1/2}\sum_{j=1}^{N} U_j \int_0^\infty \mathbb{1}_{\{z \le f(X_{t-}^N)\}} d\pi^j(t,z)$$

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Control of moment of $\sup |X_t^N|^2$ **:** if f, b bounded,

$$\mathbb{E}_{\mathcal{E}}\left[\sup_{s\le t} |X_s^N|^2\right] \le C\left(t^2 + t\left(N^{-1}\sum_{j=1}^N U_j^2\right) + t^2\left(N^{-1/2}\sum_{j=1}^N U_j\right)^2\right)$$

Main result : annealed convergence

Theorem

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- X^N converges to $\bar{X}$ in distribution in $D(\mathbb{R}_+, \mathbb{R})$
- For all $t > 0, g \in C_b^3(\mathbb{R})$,

$$\left| \mathbb{E}\left[g(X_t^N) \right] - \mathbb{E}\left[g(\bar{X}_t) \right] \right| \leq C_{g,t} \left(\frac{\ln N}{\sqrt{N}} + d_{KR}(\nu_0^N, \bar{\nu}_0) \right)$$

with $\nu_0^N := \mathcal{L}(X_0^N)$ and $\bar{\nu}_0 := \mathcal{L}(\bar{X}_0)$

Remarks :

- d_{KR} = Kantorovich-Rubinstein = 1st order Wasserstein

Main result : annealed convergence

Theorem

- X^N converges to $\bar{X}$ in distribution in $D(\mathbb{R}_+, \mathbb{R})$
- For all $t > 0, g \in C_b^3(\mathbb{R})$,

$$\left| \mathbb{E}\left[g(X_t^N) \right] - \mathbb{E}\left[g(\bar{X}_t) \right] \right| \leq C_{g,t} \left(\frac{\ln N}{\sqrt{N}} + d_{KR}(\nu_0^N, \bar{\nu}_0) \right)$$

with $\nu_0^N := \mathcal{L}(X_0^N)$ and $\bar{\nu}_0 := \mathcal{L}(\bar{X}_0)$

Remarks :

- d_{KR} = Kantorovich-Rubinstein = 1st order Wasserstein
- same convergence speed for FIDI distribution

Assumptions and example

Assumptions :

- $b, f, \sqrt{f}$ are C^4
- for $1 \leq k \leq 4$, $b^{(k)}, f^{(k)}, \sqrt{f}^{(k)}$ are bounded

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$b(x) := -\alpha x$ and $f(x) := \dfrac{\theta}{1 + e^{-\lambda(x - x_0)}}$ [Velichko, Boriskov (2020)]

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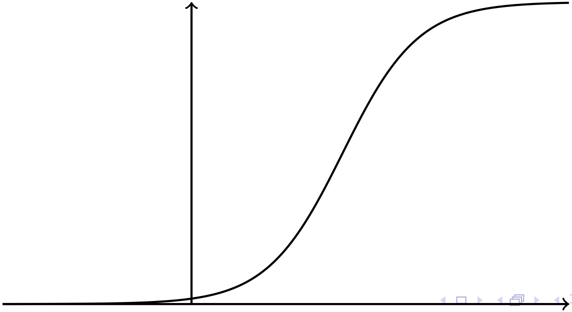

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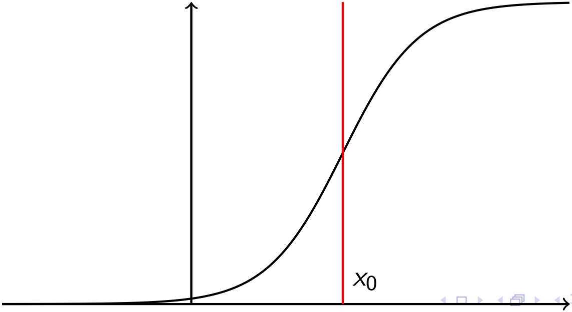

Infinitesimal generator

$N-$particle system

$$dX_t^N = b(X_t^N)dt + N^{-1/2}\sum_{j=1}^{N} U_j \int_0^\infty \mathbb{1}_{\{z\leq f(X_{t-}^N)\}} d\pi^j(t,z)$$

Limit system

$$d\bar{X}_t^N = b(\bar{X}_t^N)dt + W^{[N]} f(\bar{X}_t^N)dt + \sigma\sqrt{f(\bar{X}_t^N)}dB_t$$

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$$A_{\mathcal{E}}^N g(x) = b(x)g'(x) + f(x)\sum_{j=1}^N \left[g\left(x + \frac{U_j}{\sqrt{N}}\right) - g(x)\right]$$

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Difference of generators

$$\left|A_{\mathcal{E}}^N g(x) - \bar{A}_{\mathcal{E}}^N g(x)\right| \leq f(x) \left(\left| \frac{1}{\sqrt{N}} \sum_{j=1}^{N} U_j - W^{[N]} \right| \cdot |g'(x)| + \frac{1}{2} \left| \frac{1}{N} \sum_{j=1}^{N} U_j^2 - \sigma^2 \right| \cdot |g''(x)| + \frac{1}{6N\sqrt{N}} \sum_{j=1}^{N} |U_j|^3 \cdot ||g'''||_\infty \right)$$

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Difference of generators

$$\left|A_{\mathcal{E}}^N g(x) - \bar{A}_{\mathcal{E}}^N g(x)\right| \leq f(x) ||g||_{3,\infty} \left(\left| \frac{1}{\sqrt{N}} \sum_{j=1}^{N} U_j - W^{[N]} \right| + \frac{1}{2} \left| \frac{1}{N} \sum_{j=1}^{N} U_j^2 - \sigma^2 \right| + \frac{1}{6N\sqrt{N}} \sum_{j=1}^{N} |U_j|^3 \right)$$

Difference of generators

$$
\begin{aligned}
\left|A_{\mathcal{E}}^N g(x) - \bar{A}_{\mathcal{E}}^N g(x)\right| \leq & f(x)||g||_{3,\infty} \left(\left| \frac{1}{\sqrt{N}} \sum_{j=1}^N U_j - W^{[N]} \right| \right. \\
& + \frac{1}{2} \left| \frac{1}{N} \sum_{j=1}^N U_j^2 - \sigma^2 \right| \\
& \left. + \frac{1}{6N\sqrt{N}} \sum_{j=1}^N |U_j|^3 \right) \\
\leq & ||f||_\infty ||g||_{3,\infty} \epsilon_N(\mathcal{E})
\end{aligned}
$$

Semigroup

$$\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x) = \int_0^t P^N_{\mathcal{E},t-s}\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right) \bar{P}^N_{\mathcal{E},s} g(x) ds$$

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$$\boxed{\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x) = \int_0^t P^N_{\mathcal{E},t-s}\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right) \bar{P}^N_{\mathcal{E},s} g(x) ds}$$

Sketch of proof : $u(s) := P^N_{\mathcal{E},t-s} \bar{P}^N_{\mathcal{E},s} g(x)$

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Semigroup

$$\boxed{\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x) = \int_0^t P^N_{\mathcal{E},t-s}\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right)\bar{P}^N_{\mathcal{E},s} g(x) ds}$$

Sketch of proof : $u(s) := P^N_{\mathcal{E},t-s}\bar{P}^N_{\mathcal{E},s} g(x)$

$$\begin{aligned}\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x) =& u(t) - u(0) \\ =& \int_0^t u'(s) ds \\ =& \int_0^t \left[-\frac{d}{du}\left(P^N_{\mathcal{E},u}\bar{P}^N_{\mathcal{E},s} g(x)\right)\Big|_{u=t-s} + \frac{d}{du}\left(P^N_{\mathcal{E},t-s}\bar{P}^N_{\mathcal{E},u} g(x)\right)\Big|_{u=s}\right] ds\end{aligned}$$

Semigroup

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Semigroup convergence

$$\left| \left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t} \right) g(x) \right| \leq \int_0^t \left| P^N_{\mathcal{E},t-s} \left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}} \right) \bar{P}^N_{\mathcal{E},s} g(x) \right| ds$$

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Control of $||\bar{P}^N_{\mathcal{E},s} g(x)||_{3,\infty}$

$$\bar{P}^N_{\mathcal{E},s} g(x) = \mathbb{E}_{\mathcal{E},\bar{X}_0^N=x}\left[g(\bar{X}^N_s)\right]$$

Semigroup convergence

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Semigroup convergence

$$\left|\left(\bar{P}^N_{\mathcal{E},t} - P^N_{\mathcal{E},t}\right) g(x)\right| \leq \int_0^t \left| P^N_{\mathcal{E},t-s}\left(\bar{A}^N_{\mathcal{E}} - A^N_{\mathcal{E}}\right) \bar{P}^N_{\mathcal{E},s} g(x)\right| ds$$

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Semigroup convergence

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$$|\partial_x \bar{P}^N_{\mathcal{E},s} g(x)| \leq ||g'||_\infty \sup_x \mathbb{E}_{\mathcal{E}}\left[|\partial_x \bar{X}^N_s(x)|\right] \leq C_s ||g'||_\infty e^{C_s |W^{[N]}|}$$

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$$\leq ||f||_\infty \left(\int_0^t \left|\left|\bar{P}^N_{\mathcal{E},s} g\right|\right|_{3,\infty} ds\right)\epsilon_N(\mathcal{E})$$
$$\leq C_t e^{C_t|W^{[N]}|}\epsilon_N(\mathcal{E})$$

Control of $||\bar{P}^N_{\mathcal{E},s} g(x)||_{3,\infty}$

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One-dimensional convergence

$$\left| \mathbb{E}_{\mathcal{E}} \left[g(X_t^N) \right] - \mathbb{E}_{\mathcal{E}} \left[g(\bar{X}_t^N) \right] \right|$$

One-dimensional convergence

$$\left|\mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right]-\mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right]\right|$$
$$=\left|\int d\nu_0^N(x)P_{\mathcal{E},t}^N g(x)-\int d\bar{\nu}_0(x)\bar{P}_{\mathcal{E},t}^N g(x)\right|$$

One-dimensional convergence

$$\left|\mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right]\right|$$

$$= \left|\int d\nu_0^N(x) P_{\mathcal{E},t}^N g(x) - \int d\bar{\nu}_0(x) \bar{P}_{\mathcal{E},t}^N g(x)\right|$$

$$\leq \int d\nu_0^N(x) \left|\left(P_{\mathcal{E},t}^N - \bar{P}_{\mathcal{E},t}^N\right) g(x)\right| + \left|\int d(\nu_0^N - \bar{\nu}_0)(x) \bar{P}_{\mathcal{E},t}^N g(x)\right|$$

One-dimensional convergence

$$\left|\mathbb{E}_{\mathcal{E}}\left[g(X_t^N)\right] - \mathbb{E}_{\mathcal{E}}\left[g(\bar{X}_t^N)\right]\right|$$

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One-dimensional convergence

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$$\mathbb{E}\left[(X_t^N - X_s^N)^2 (X_s^N - X_r^N)^2\right] \leq C_T (t-r)^{3/2}$$

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Model in dimension N

Random environment iid U_{ji} =synaptic strength $j \to i$

$$dX_t^{N,i} = b(X_t^{N,i})dt + N^{-1/2} \sum_{j=1}^{N} U_{ji} \int_0^\infty \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z)$$

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Problem : not explicit, not (conditional) McKean-Vlasov

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Girsanov's theorem gives $\mathcal{L}((X^{N,i})_{1 \leq i \leq N}) / \mathcal{L}((\tilde{X}^{N,i})_{1 \leq i \leq N})$

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Thank you for your attention !

Questions ?