Conditional propagation of chaos for mean field system of interacting neurons

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joint work with Eva Löcherbach[2] and Dasha Loukianova[3]

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1. Introduction
   - Model
   - Discussion about the hypothesis
2. Limit system
   - Heuristics
   - Conditional propagation of chaos

Point process : definitions

Point process Z :

- a random countable set of $\mathbb{R}_+$: $Z = \{T_i : i \in \mathbb{N}\}$
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$$\forall 0 \leq a < b, \mathbb{E}\left[Z([a,b])|\mathcal{F}_a\right] = \mathbb{E}\left[\int_a^b \lambda_t dt \middle| \mathcal{F}_a\right]$$

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Marked point process $Z = \{(T_i, U_i) : i \in \mathbb{N}\}$ (U_i iid)

Notation abuse $U_i =: U(T_i)$

Modeling in neuroscience

Neural activity $=$ Set of spike times

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Network of N neurons :

$$\begin{aligned} Z^{N,i} &= \text{set of spike times of neuron } i \\ &= \text{point process with intensity } f(X^{N,i}_{t-}) \end{aligned}$$

$$X^{N,i} = \text{potential of neuron } i$$

$N-$neurons network model

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^{N} U^j(t) dZ_t^{N,j} - X_{t-}^{N,i} dZ_t^{N,i}$$

with :

- $Z^{N,j}$ = marked point processes with intensity $f(X_{t-}^{N,j})$
- $U^j(t)$ = mark of t (if t atom of $Z^{N,j}$)
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- ν is centered, $\int |u|^3 d\nu(u) < \infty$ and $\sigma^2 := \int u^2 d\nu(u)$

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- $X_t^{N,i} = 0$ if neuron i emits a spike at t
  $\rightarrow$ repolarization

$N-$neurons network dynamics

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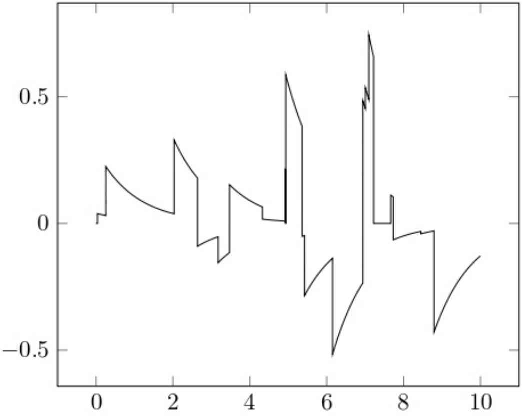

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Scaling N^{-1} or $N^{-1/2}$:

- N^{-1} (LLN) $\Longrightarrow$ limit ODE
- $N^{-1/2}$ (CLT) $\Longrightarrow$ limit SDE ([Barral, D Reyes (2016)])

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Centered $U^j(t)$: "balanced networks"
$\to$ excitatory/inhibitory inputs are balanced
([Shu, Hasenstaub, McCormick (2003)], [Haider et al. (2006)])

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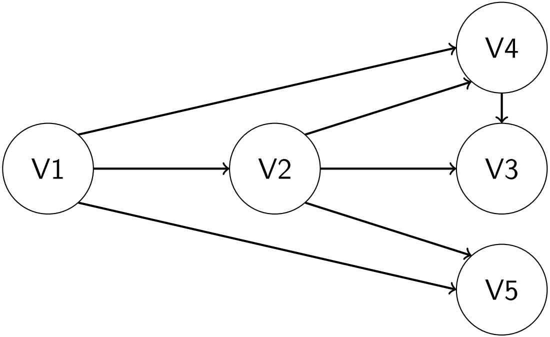
Multi-population model : network divided into K complete graphs
Example : visual cortex divided into 5 layers V1-V5 ([Hubel (1995)])



Limit system : heuristic (1)

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with :

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- $\bar{Z}^i$ point process with intensity $f(\bar{X}_t^i)$

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Then $\bar{M}$ should satisfy

$$\bar{M}_t = \sigma \int_0^t \sqrt{\lim_N \frac{1}{N} \sum_{j=1}^{N} f(\bar{X}_s^j)} dW_s = \sigma \int_0^t \sqrt{\lim_N \bar{\mu}_s^N(f)} dW_s$$

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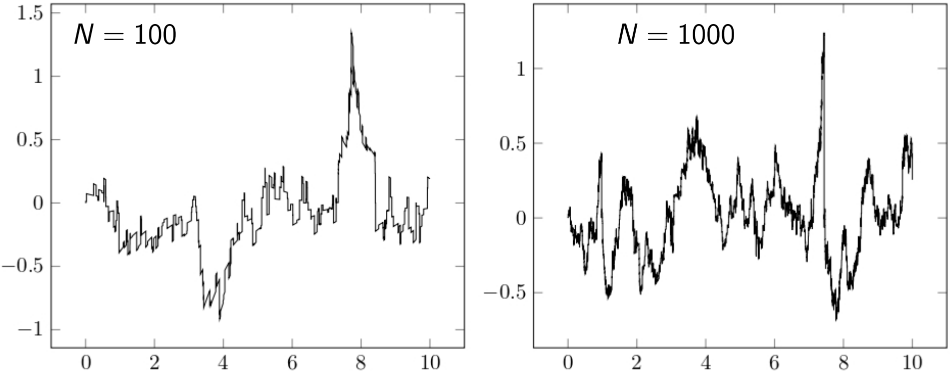
Simulations of $X^{N,1}$



Convergence of $(X^{N,i})_{1\leq i\leq N}$

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Result [E., Löcherbach, Loukianova (2021a)]

$(X^{N,i})_{1 \leq i \leq N}$ converges to $(\bar{X}^i)_{i \geq 1}$ in $D^{\mathbb{N}^*}$ in law

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NSC : $\mu^N := \sum_{j=1}^N \delta_{X^{N,j}}$ converges to $\bar{\mu} := \mathcal{L}(\bar{X}^1|W)$ in $\mathcal{P}(D)$

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Outline of the proof

Step 1. $(\mu^N)_N$ is tight on $\mathcal{P}(D)$ (i.e. $(\mathcal{L}(\mu^N))_N$ is relatively compact)

Equivalent condition : $(X^{N,1})_N$ is tight on D

Proof : Aldous' criterion

Step 2. Identifying the limit distribution of $(\mu^N)_N$

Proof : any limit of μ^N is solution of a martingale problem

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$$P(A \times B) := \int_{\mathcal{P}(D)} \mathbb{1}_A(m) m \otimes m(B) dQ(m)$$

Q is solution of $(\mathcal{M})$ if for all $g \in C_b^2(\mathbb{R}^2)$,
$g(\bar{X}_t^1, \bar{X}_t^2) - g(\bar{X}_0^1, \bar{X}_0^2) - \int_0^t \bar{L}g(\bar{\mu}_s, \bar{X}_s^1, \bar{X}_s^2) ds$ is a martingale

$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + \sigma \sqrt{\bar{\mu}_t(f)} dW_t - \bar{X}_{t-}^i d\bar{Z}_t^i$$

$$\bar{L}g(m, x^1, x^2) = -\alpha x^1 \partial_1 g(x) - \alpha x^2 \partial_2 g(x) + \frac{\sigma^2}{2} m(f) \sum_{i,j=1}^{2} \partial_{i,j}^2 g(x)$$
$$+ f(x^1)(g(0, x^2) - g(x)) + f(x^2)(g(x^1, 0) - g(x))$$

Convergence of μ^N to the solution of $(\mathcal{M})$

$$dX_t^{N,i} = -\alpha X_t^{N,i} + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j\neq i}}^{N} U^j(t) dZ_t^{N,j} - X_{t-}^{N,i} dZ_t^{N,i}$$

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$$L^N g(m, x^1, x^2) = -\alpha x^1 \partial_1 g(x) - \alpha x^2 \partial_2 g(x)$$

$$+N \cdot m(f) \int \left[g(x^1 + u \cdot N^{-1/2}, x^2 + u \cdot N^{-1/2}) - g(x) \right] d\nu(u)$$

$$+f(x^1) \int (g(0, x^2 + u \cdot N^{-1/2}) - g(x)) d\nu(u)$$

$$+f(x^2) \int (g(x^1 + u \cdot N^{-1/2}, 0) - g(x)) d\nu(u)$$

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$$\left| \mathbb{E} \left[\int \mu^N \otimes \mu^N (dx) \bar{L} g(\mu_t^N, x_t^1, x_t^2) - \int \mu^N \otimes \mu^N (dx) L^N g(\mu_t^N, x_t^1, x_t^2) \right] \right|$$

Convergence of μ^N to the solution of $(\mathcal{M})$

$$dX_t^{N,i} = -\alpha X_t^{N,i} + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j\neq i}}^{N} U^j(t) dZ_t^{N,j} - X_{t-}^{N,i} dZ_t^{N,i}$$

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$$+ f(x^2) \int (g(x^1 + u \cdot N^{-1/2}, 0) - g(x)) d\nu(u)$$

Taylor-Lagrange's inequality :

$$\left| \mathbb{E} \left[\int \mu^N \otimes \mu^N(dx) \bar{L} g(\mu_t^N, x_t^1, x_t^2) - \int \mu^N \otimes \mu^N(dx) L^N g(\mu_t^N, x_t^1, x_t^2) \right] \right|$$
$$\leq C_t \cdot N^{-1/2} \underset{N\to\infty}{\longrightarrow} 0$$

Convergence of $(\mu^N)_N$

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^{N} U^j(t) dZ_t^{N,j} - X_{t-}^{N,i} dZ_t^{N,i}$$

$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + \sigma \sqrt{\bar{\mu}_t(f)} dW_t - \bar{X}_{t-}^i d\bar{Z}_t^i$$

Main steps of the proof :

- $(\mathcal{L}(\mu^N))_N$ relatively compact
- the only limit is (the unique) solution of $(\mathcal{M})$
- $\Rightarrow (\mu^N)_N$ converges (in law) to $\mathcal{L}(\bar{X}^1 | W)$

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Thank you for your attention !

Questions ?