

Conditional propagation of chaos for mean field systems of interacting neurons

Xavier Erny ¹

Joint work with: Eva Löcherbach ² and Dasha Loukianova ¹

¹Université d'Evry (LaMME)

²Université Paris 1 Panthéon-Sorbonne (SAMM)

Groupe de travail PEIPS, 12 novembre 2020

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 - Model
 - Limit system

Modeling in neuroscience

Neural activity = Set of spike times

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Here, $X^{N,i}$ solves an SDE directed by $(Z^{N,j})_{1 \leq j \leq N}$

Mean field limit

N -particle system :

- $Z_t^{N,i} = \int_0^t \int_0^\infty 1_{\{z \leq f(X_{s-}^{N,i})\}} d\pi^i(s, z)$

- $dX_t^{N,i} = b(X_t^{N,i})dt + \sum_{j=1}^N \int_0^\infty u^{ji}(t) 1_{\{z \leq f(X_{t-}^{N,j})\}} d\pi^j(t, z)$

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- linear scaling N^{-1} (LLN) :
[\[Delattre et al. \(2016\)\]](#) (Hawkes process, $u^{ji}(t) = 1$),
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- diffusive scaling $N^{-1/2}$ (CLT) :
[\[E. et al. \(2019\)\]](#) random and centered $u^{ji}(s)$

Linear scaling

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^N \int_0^\infty 1_{\{z \leq f(X_{t-}^{N,j})\}} d\pi^j(t, z) \\ - \int_0^\infty \int_{\mathbb{R}} X_{t-}^{N,i} 1_{\{z \leq f(X_{t-}^{N,i})\}} d\pi^i(t, z, u)$$

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Intepretation :

- drift : $-\alpha x$ models an exponential loss of the potential
- small jump of order N^{-1} : the effect of spike of one neuron to the potential of the others
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[De Masi et al. (2015)] and [Fournier & Löcherbach (2016)]

Generalization to McKean-Vlasov frame [Andreis et al. (2018)]

Diffusive scaling

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^N \int_0^\infty \int_{\mathbb{R}} u \mathbf{1}_{\{z \leq f(X_{t-}^{N,j})\}} d\pi^j(t, z, u) \\ - \int_0^\infty \int_{\mathbb{R}} X_{t-}^{N,i} \mathbf{1}_{\{z \leq f(X_{t-}^{N,i})\}} d\pi^i(t, z, u)$$

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ν probability measure on \mathbb{R} centered with $\int_{\mathbb{R}} |u|^3 d\nu(u) < \infty$

$$\sigma^2 = \int_{\mathbb{R}} u^2 d\nu(u)$$

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Dynamic of $X^{N,i}$:

- $X_t^{N,i} = X_s^{N,i} e^{-\alpha(t-s)}$ if the system does not jump in $[s, t]$
- $X_t^{N,i} = X_{t-}^{N,i} + \frac{U}{\sqrt{N}}$ if a neuron $j \neq i$ emits a spike at t
- $X_t^{N,i} = 0$ if neuron i emits a spike at t

Limit system : heuristic (1)

$$\begin{aligned}
 dX_t^{N,i} = & -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^N \int_{\mathbb{R}_+ \times \mathbb{R}} u \mathbf{1}_{\{z \leq f(X_{t-}^{N,j})\}} d\pi^j(t, z, u) \\
 & - X_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbf{1}_{\{z \leq f(X_{t-}^{N,i})\}} d\pi^i(t, z, u)
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$$M_t^N := \frac{1}{\sqrt{N}} \sum_{j=1}^N \int_{[0,t] \times \mathbb{R}_+ \times \mathbb{R}} u \mathbf{1}_{\{z \leq f(X_{s-}^{N,j})\}} d\pi^j(s, z, u)$$

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$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + d\bar{M}_t \\ - \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbf{1}_{\{z \leq f(\bar{X}_{t-}^i)\}} d\pi^i(t, z, u)$$

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$$\langle \bar{M} \rangle_t = \lim_N \langle M^N \rangle_t = \lim_N \sigma^2 \int_0^t \frac{1}{N} \sum_{j=1}^N f(X_s^{N,j}) ds$$

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Then \bar{M} should satisfy

$$\bar{M}_t = \sigma \int_0^t \sqrt{\lim_N \frac{1}{N} \sum_{j=1}^N f(\bar{X}_s^j)} dW_s = \sigma \int_0^t \sqrt{\lim_N \bar{\mu}_s^N(f)} dW_s$$

with $\bar{\mu}^N := \frac{1}{N} \sum_{j=1}^N \delta_{\bar{X}^j}$

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μ is the limit of empirical measures of $(\bar{X}^i)_{i \geq 1}$ exchangeable by Proposition (7.20) of [Aldous (1983)] μ is the directing measure of $(\bar{X}^i)_{i \geq 1}$ (conditionally on μ , \bar{X}^i i.i.d. $\sim \mu$)

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Well-posedness of the limit SDE

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Problems :

- conditional expectation in the Brownian term (McKean-Vlasov frame)
- unbounded jumps (non-Lipschitz compensator $x \rightarrow xf(x)$)
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Assumptions :

- $\inf f > 0, \sup f < \infty$
- f is Lipschitz continuous
- $\mathbb{E}[e^{a|\bar{X}_0}|] < \infty$

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To prove trajectorial uniqueness :

- $u(t) = \mathbb{E}[|\hat{X}_t^i - \check{X}_t^i|]$ (problem with Brownian term)
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Idea of [Graham \(1992\)](#) : $u(t) = \mathbb{E} \left[\sup_{0 \leq s \leq t} |\hat{X}_s^i - \check{X}_s^i| \right]$

Drift and Brownian terms

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$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq s \leq t} \left| \int_0^s (\sqrt{\hat{\mu}_r(f)} - \sqrt{\check{\mu}_r(f)}) dW_r \right| \right] \\ & \leq \mathbb{E} \left[\left(\int_0^t (\sqrt{\hat{\mu}_s(f)} - \sqrt{\check{\mu}_s(f)})^2 ds \right)^{1/2} \right] \\ & \leq C \mathbb{E} \left[\left(\int_0^t (\hat{\mu}_s(f) - \check{\mu}_s(f))^2 ds \right)^{1/2} \right] \\ & \leq C\sqrt{t} \mathbb{E} \left[\sup_{0 \leq s \leq t} \left| \mathbb{E} [f(\hat{X}_s) | W] - \mathbb{E} [f(\check{X}_s) | W] \right| \right] \leq C\sqrt{t}u(t) \end{aligned}$$

Jump term

$$\mathbb{E} \left[\sup_{0 \leq s \leq t} \dots \right] \leq \int_0^t \mathbb{E} \left[|\hat{X}_s - \check{X}_s| f(\check{X}_s) \vee f(\hat{X}_s) \right] ds$$

$$+ \int_0^t (|\hat{X}_s| + |\check{X}_s|) |f(\hat{X}_s) - f(\check{X}_s)| ds$$

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$$\begin{aligned}
\mathbb{E} \left[\sup_{0 \leq s \leq t} \dots \right] &\leq \int_0^t \mathbb{E} \left[|\hat{X}_s - \check{X}_s| f(\check{X}_s) \vee f(\hat{X}_s) \right] ds \\
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&\leq C \int_0^t u(s) ds + C \int_0^t r_s u(s) ds \\
&\quad + C \int_0^t \left(\mathbb{E} \left[|\hat{X}_s| \mathbf{1}_{\{|\hat{X}_s| > r_s\}} \right] + \mathbb{E} \left[|\check{X}_s| \mathbf{1}_{\{|\check{X}_s| > r_s\}} \right] \right) ds
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&\leq C \int_0^t \left[(1 + r_s) u(s) + e^{-r_s} \right] ds
\end{aligned}$$

Uniqueness for the limit SDE

Combining the inequalities : for all $0 \leq t \leq T$,

$$u(t) \leq C(T + \sqrt{T})u(t) + C \int_0^t [(1 + r_s)u(s) + e^{-r_s}] ds$$

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Existence for the limit SDE

$$\begin{aligned}dX_t^{[n+1]} &= -\alpha X_t^{[n+1]} dt + \sigma \sqrt{\mu_t^{[n]}(f)} dW_t \\ &\quad - \int_0^\infty \int_{\mathbb{R}} X_{t-}^{[n+1]} \mathbf{1}_{\{z \leq f(X_{t-}^{[n]})\}} d\pi(t, z, u)\end{aligned}$$

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Another version of the limit system

The strong limit system :

$$\begin{aligned}d\bar{X}_t^i &= -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mathbb{E}[f(\bar{X}_t^i) | W]} dW_t \\ &\quad - \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbf{1}_{\{z \leq f(\bar{X}_{t-}^i)\}} d\pi^i(t, z, u)\end{aligned}$$

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$$\begin{aligned}d\bar{Y}_t^i &= -\alpha \bar{Y}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t \\ &\quad - \bar{Y}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbf{1}_{\{z \leq f(\bar{Y}_{t-}^i)\}} d\pi^i(t, z, u)\end{aligned}$$

where $\mu_t = \mathcal{L}(\bar{Y}_t^1 | \mu_t)$ is the directing measure of $(\bar{Y}_t^i)_{i \geq 1}$

Equivalence between the two systems

An auxiliary system :

$$d\tilde{X}_t^{N,i} = -\alpha \tilde{X}_t^{N,i} dt + \sigma \sqrt{\frac{1}{N} \sum_{j=1}^N f(\tilde{X}_t^{N,j})} dW_t$$

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By Osgood's lemma,

$$u_N(t) \leq CN^{-C/2} \xrightarrow{N \rightarrow \infty} 0$$

Convergence of $(X^{N,i})_{1 \leq i \leq N}$

$$\begin{aligned}dX_t^{N,i} &= -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j \neq i} \int_{\mathbb{R}_+ \times \mathbb{R}} u 1_{\{z \leq f(X_{t-}^{N,j})\}} d\pi^j(t, z, u) \\ &\quad - X_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\{z \leq f(X_{t-}^{N,i})\}} d\pi^i(t, z, u) \\ d\bar{X}_t^i &= -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t \\ &\quad - \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\{z \leq f(\bar{X}_{t-}^i)\}} d\pi^i(t, z, u)\end{aligned}$$

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Equivalent condition (Proposition (7.20) of [\[Aldous \(1983\)\]](#)) :
 $\mu^N := \sum_{j=1}^N \delta_{X^{N,j}}$ converges to $\mu := \mathcal{L}(\bar{X}^1 | W)$ in $\mathcal{P}(D)$

Outline of the proof

Step 1. $(\mu^N)_N$ is tight on $\mathcal{P}(D)$

Equivalent condition : $(X^{N,1})_N$ is tight on D

Proof : Aldous' criterion

Step 2. Identifying the limit distribution of $(\mu^N)_N$

Proof : any limit of μ^N is solution of a martingale problem

Martingale problem

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$$\begin{aligned} Lg(m, x^1, x^2) = & -\alpha x^1 \partial_1 g(x) - \alpha x^2 \partial_2 g(x) + \frac{\sigma^2}{2} m(f) \sum_{i,j=1}^2 \partial_{i,j}^2 g(x) \\ & + f(x^1)(g(0, x^2) - g(x)) + f(x^2)(g(x^1, 0) - g(x)) \end{aligned}$$

Uniqueness for the martingale problem

Let Q be a solution of (\mathcal{M}) . Write $Q = \mathcal{L}(\mu)$ where μ is the directing measure of some exchangeable system $(\bar{Y}^i)_{i \geq 1}$

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Representation theorems imply (admitted)

$$\begin{aligned} \forall i \in \{1, 2\}, d\bar{Y}_t^i &= -\alpha \bar{Y}_t^i dt + \sqrt{\mu_t(f)} dW_t \\ &\quad - \bar{Y}_{t-}^i \int_{\mathbb{R}_+} 1_{\{z \leq f(\bar{Y}_{t-}^i)\}} d\pi^i(t, z) \end{aligned}$$

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Then the law of $\mu = \mathcal{L}(\bar{Y}^1 | W)$ is uniquely determined

Convergence of μ^N to the solution of (\mathcal{M})

Let μ be the limit of (a subsequence of) μ^N

$\mathcal{L}(\mu)$ is solution of (\mathcal{M}) if

$$\mathbb{E}[F(\mu)] = 0$$

for any F of the form

$$\begin{aligned} F(m) := & \int_{D^2} m \otimes m(d\gamma) \phi_1(\gamma_{s_1}) \dots \phi_k(\gamma_{s_k}) \left[\phi(\gamma_t) - \phi(\gamma_s) \right. \\ & + \alpha \int_s^t \gamma_r^1 \partial_1(\gamma_r) dr + \alpha \int_s^t \gamma_r^2 \partial_2(\gamma_r) dr \\ & - \int_s^t f(\gamma_r^1) (\phi(0, \gamma_r^2) - \phi(\gamma_r)) dr - \int_s^t f(\gamma_r^2) (\phi(\gamma_r^1, 0) - \phi(\gamma_r)) dr \\ & \left. - \frac{\sigma^2}{2} \int_s^t m_r(f) \sum_{i_1, i_2=1}^2 \partial_{i_1, i_2}^2 \phi(\gamma_r) dr \right] \end{aligned}$$

The expression of $F(\mu^N)$

$$\begin{aligned}
 F(\mu^N) := & \int_{D^2} \mu^N \otimes \mu^N (d\gamma) \phi_1(\gamma_{s_1}) \dots \phi_k(\gamma_{s_k}) \left[\phi(\gamma_t) - \phi(\gamma_s) \right. \\
 & + \alpha \int_s^t \gamma_r^1 \partial_1(\gamma_r) dr + \alpha \int_s^t \gamma_r^2 \partial_2(\gamma_r) dr \\
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 & \left. - \int_s^t f(\gamma_r^2) (\phi(\gamma_r^1, 0) - \phi(\gamma_r)) dr \right. \\
 & \left. - \frac{\sigma^2}{2} \int_s^t \mu_r^N(f) \sum_{i_1, i_2=1}^2 \partial_{i_1, i_2}^2 \phi(\gamma_r) dr \right]
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The expression of $F(\mu^N)$

$$\begin{aligned}
 F(\mu^N) := & \int_{D^2} \mu^N \otimes \mu^N(d\gamma) \phi_1(\gamma_{s_1}) \dots \phi_k(\gamma_{s_k}) \left[\phi(\gamma_t) - \phi(\gamma_s) \right. \\
 & + \alpha \int_s^t \gamma_r^1 \partial_1(\gamma_r) dr + \alpha \int_s^t \gamma_r^2 \partial_2(\gamma_r) dr \\
 & - \int_s^t f(\gamma_r^1) (\phi(0, \gamma_r^2) - \phi(\gamma_r)) dr \\
 & \left. - \int_s^t f(\gamma_r^2) (\phi(\gamma_r^1, 0) - \phi(\gamma_r)) dr \right. \\
 & \left. - \frac{\sigma^2}{2} \int_s^t \mu_r^N(f) \sum_{i_1, i_2=1}^2 \partial_{i_1, i_2}^2 \phi(\gamma_r) dr \right]
 \end{aligned}$$

The expression of $F(\mu^N)$

$$\begin{aligned}
 F(\mu^N) := & \\
 & \frac{1}{N^2} \sum_{i,j=1}^N \phi_1(X_{s_1}^{N,i}, X_{s_1}^{N,j}) \dots \phi_k(X_{s_k}^{N,i}, X_{s_k}^{N,j}) \left[\phi(X_t^{N,i}, X_t^{N,j}) - \phi(X_s^{N,i}, X_s^{N,j}) \right. \\
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 \end{aligned}$$

The expression of $\phi(X^{N,i}, X^{N,j})$

By Ito's formula,

$$\begin{aligned} & \mathbb{E} \phi(X_t^{N,i}, X_t^{N,j}) - \phi(X_s^{N,i}, X_s^{N,j}) = \\ & \mathbb{E} -\alpha \int_s^t X_r^{N,i} \partial_1(X_r^{N,i}, X_r^{N,j}) dr - \alpha \int_s^t X_r^{N,j} \partial_2(X_r^{N,i}, X_r^{N,j}) dr \\ & + \int_s^t \int_{\mathbb{R}} f(X_r^{N,i}) \left(\phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j}) \right) d\nu(u) dr \\ & + \int_s^t \int_{\mathbb{R}} f(X_r^{N,j}) \left(\phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, 0) - \phi(X_r^{N,i}, X_r^{N,j}) \right) d\nu(u) dr \\ & + \int_s^t \int_{\mathbb{R}} \sum_{\substack{k=1 \\ k \neq i,j}}^N f(X_r^{N,k}) \left(\phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j}) \right) d\nu(u) dr \end{aligned}$$

Vanishing of $\mathbb{E} [F(\mu^N)]$

The reset jump term

$$\left| \phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}}) \right|$$

Vanishing of $\mathbb{E} [F(\mu^N)]$

The reset jump term

$$\left| \phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}}) \right| \leq C \frac{|u|}{\sqrt{N}}$$

Vanishing of $\mathbb{E} [F(\mu^N)]$

The **reset jump term**

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The **small jump term**

$$N \left| \phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j}) - \frac{u^2}{2N} \sum_{i_1, i_2=1}^2 \partial_{i_1, i_2}^2 \phi(X_r^{N,i}, X_r^{N,j}) \right|$$

Vanishing of $\mathbb{E} [F(\mu^N)]$

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$$CN^{-1/2} \geq \mathbb{E} [F(\mu^N)] \xrightarrow{N \rightarrow \infty} \mathbb{E} [F(\mu)] = 0$$

Convergence of $(\mu^N)_N$

$$\begin{aligned}dX_t^{N,i} &= -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j \neq i} \int_{\mathbb{R}_+ \times \mathbb{R}} u \mathbf{1}_{\{z \leq f(X_{t-}^{N,j})\}} d\pi^j(t, z, u) \\ &\quad - X_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbf{1}_{\{z \leq f(X_{t-}^{N,i})\}} d\pi^i(t, z, u) \\ d\bar{X}_t^i &= -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t \\ &\quad - \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbf{1}_{\{z \leq f(\bar{X}_{t-}^i)\}} d\pi^i(t, z, u)\end{aligned}$$

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- $(\mu^N)_N$ is tight on $\mathcal{P}(D)$

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- $(\mu^N)_N$ is tight on $\mathcal{P}(D)$
- let μ be the limit of a converging subsequence
- $\mathcal{L}(\mu)$ is the unique solution of (\mathcal{M})
- $\mu = \mathcal{L}(\bar{X}^1 | W)$ is the only limit of $(\mu^N)_N$

McKean-Vlasov model

$$dX_t^{N,i} = b(X_t^{N,i}, \mu_t^N)dt + \sigma(X_t^{N,i}, \mu_t^N)d\beta_t^i \\ + \frac{1}{\sqrt{N}} \sum_{k=1}^N \int_{\mathbb{R}_+ \times \mathbb{R}^{N^*}} \Psi(X_{t-}^{N,k}, X_{t-}^{N,i}, \mu_{t-}^N, u^k, u^i) \mathbf{1}_{\{z \leq f(X_{t-}^{N,k}, \mu_{t-}^N)\}} d\pi^k(t, z, u)$$

with $\mu^N := \frac{1}{N} \sum_{i=1}^N \delta_{X^{N,i}}$, and π^k has intensity $dt \cdot dz \cdot \nu(du)$
($\nu = \nu_1^{\otimes N^*}$)

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Dynamic of $X^{N,i}$:

- while there is no jump, the dynamic is given by the drift and Brownian terms
- if there is a jump at time t , created by neuron k , each neuron i creates a r.v. U^i (the U^i are i.i.d.),

$$X_t^{N,i} = X_{t-}^{N,i} + \frac{1}{\sqrt{N}} \Psi(X_{t-}^{N,k}, X_{t-}^{N,i}, \mu_{t-}^N, U^k, U^i)$$

Heuristics for the limit system

$$dX_t^{N,i} = b(X_t^{N,i}, \mu_t^N)dt + \sigma(X_t^{N,i}, \mu_t^N)d\beta_t^i \\ + \frac{1}{\sqrt{N}} \sum_{k=1}^N \int_{\mathbb{R}_+ \times \mathbb{R}^{N^*}} \Psi(X_{t-}^{N,k}, X_{t-}^{N,i}, \mu_{t-}^N, u^k, u^i) \mathbf{1}_{\{z \leq f(X_{t-}^{N,k}, \mu_{t-}^N)\}} d\pi^k(t, z, u)$$

Heuristics for the limit system

$$dX_t^{N,i} = b(X_t^{N,i}, \mu_t^N)dt + \sigma(X_t^{N,i}, \mu_t^N)d\beta_t^i \\ + \frac{1}{\sqrt{N}} \sum_{k=1}^N \int_{\mathbb{R}_+ \times \mathbb{R}^{N^*}} \Psi(X_{t-}^{N,k}, X_{t-}^{N,i}, \mu_{t-}^N, u^k, u^i) 1_{\{z \leq f(X_{t-}^{N,k}, \mu_{t-}^N)\}} d\pi^k(t, z, u)$$

$$J_t^{N,i} = \frac{1}{\sqrt{N}} \sum_{k=1}^N \int \Psi(X_{s-}^{N,k}, X_{s-}^{N,i}, \mu_{s-}^N, u^k, u^i) 1_{\{z \leq f(X_{s-}^{N,k}, \mu_{s-}^N)\}} d\pi^k(s, z, u)$$

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Particular case

$$\Psi(x, y, m, u, v) = \Psi(u, v)$$

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with $\zeta^2 = \int \Psi(u^1, u^2)^2 \nu(du)$

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$$\bar{J}_t^i = \kappa \int_0^t \sqrt{\int_{\mathbb{R}} f(x, \mu_s) \mu_s(dx)} dW_s + \sqrt{\varsigma^2 - \kappa^2} \int_0^t \sqrt{\int_{\mathbb{R}} f(x, \mu_s) \mu_s(dx)} dW_s^i$$

with W, W^i i.i.d. Brownian motions and $\mu = \mathcal{L}(\bar{X}^i | W)$

General case

$$\langle J^{N,i}, J^{N,j} \rangle_t = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{N^*}} \Psi(x, X_s^{N,i}, \mu_s^N, u^1, u^i) \Psi(x, X_s^{N,j}, \mu_s^N, u^1, u^j) f(x, \mu_s^N) \nu(du) \mu_s^N(dx) ds$$

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Problem : the blue term is not a product, but an integral of a product

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Solution : let $M(dt, dz) = M_t(dz)$ be a martingale measure on $\mathbb{R}_+ \times E$ with intensity $dt \cdot m_t(dz)$,

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Here : $E = \mathbb{R}^{N^*} \times \mathbb{R}$ and $m_s(du, dx) = \nu(du) \cdot \mu_s(dx)$

Limit system (1)

$$\begin{aligned}d\bar{X}_t^i &= b(\bar{X}_t^i, \mu_t)dt + \sigma(\bar{X}_t^i, \mu_t)d\beta_t^i \\ &+ \int \tilde{\Psi}(x, \bar{X}_t^i, \mu_t, \nu)\sqrt{f(x, \mu_t)}dM(t, x, \nu) \\ &+ \int \kappa(x, \bar{X}_t^i, \mu_t)\sqrt{f(x, \mu_t)}dM^i(t, x)\end{aligned}$$

with

$$\begin{aligned}\tilde{\Psi}(x, y, m, \nu) &= \int_{\mathbb{R}^{N^*}} \Psi(x, y, m, \nu, u^1)\nu(du) \\ \kappa(x, y, m)^2 &= \int \Psi(x, y, m, u^1, u^2)^2\nu(du) - \int \tilde{\Psi}(x, y, m, u^1)^2\nu(du)\end{aligned}$$

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Interpretation of $\tilde{\Psi}$:

$$\int \tilde{\Psi}(x, y, m, u^1)^2\nu(du) = \int \Psi(x, y, m, u^1, u^2)\Psi(x, y, m, u^1, u^3)\nu(du)$$

Limit system (2)

$$\begin{aligned}d\bar{X}_t^i &= b(\bar{X}_t^i, \mu_t)dt + \sigma(\bar{X}_t^i, \mu_t)d\beta_t^i \\ &+ \int \tilde{\Psi}(x, \bar{X}_t^i, \mu_t, \nu)\sqrt{f(x, \mu_t)}dM(t, x, \nu) \\ &+ \int \kappa(x, \bar{X}_t^i, \mu_t)\sqrt{f(x, \mu_t)}dM^i(t, x)\end{aligned}$$

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M and M^i are orthogonal (not independent) :

$$\begin{aligned}M_t^i(A) &= \int_0^t \int_0^1 1_A(F_s^{-1}(p))dW^i(s, p) \\ M_t(A \times B) &= \int_0^t \int_0^1 \int_{\mathbb{R}} 1_A(F_s^{-1}(p))1_B(u)dW(s, p, u)\end{aligned}$$

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Generator of the limit system

$$\begin{aligned}d\bar{X}_t^i &= b(\bar{X}_t^i, \mu_t)dt + \sigma(\bar{X}_t^i, \mu_t)d\beta_t^i \\ &+ \int \tilde{\Psi}(x, \bar{X}_t^i, \mu_t, \nu)\sqrt{f(x, \mu_t)}dM(t, x, \nu) \\ &+ \int \kappa(x, \bar{X}_t^i, \mu_t)\sqrt{f(x, \mu_t)}dM^i(t, x)\end{aligned}$$

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$$\begin{aligned}Lg(y, m, x, \nu) &= b(y^1, m)\partial_{y^1}g(y) + b(y^2, m)\partial_{y^2}g(y) \\ &+ \frac{1}{2}\sigma(y^1, m)^2\partial_{y^1}^2g(y) + \frac{1}{2}\sigma(y^2, m)^2\partial_{y^2}^2g(y) \\ &+ \frac{1}{2}f(x, m)\kappa(x, y^1, m)^2\partial_{y^1}^2g(y) + \frac{1}{2}f(x, m)\kappa(x, y^2, m)^2\partial_{y^2}^2g(y) \\ &+ \frac{1}{2}f(x, m)\sum_{i,j=1}^2\tilde{\Psi}(x, y^i, m, \nu)\tilde{\Psi}(x, y^j, m, \nu)\partial_{y^i y^j}^2g(y)\end{aligned}$$

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$$\begin{aligned}d\bar{X}_t^i &= b(\bar{X}_t^i, \mu_t)dt + \sigma(\bar{X}_t^i, \mu_t)d\beta_t^i \\ &+ \int \tilde{\Psi}(x, \bar{X}_t^i, \mu_t, \nu)\sqrt{f(x, \mu_t)}dM(t, x, \nu) \\ &+ \int \kappa(x, \bar{X}_t^i, \mu_t)\sqrt{f(x, \mu_t)}dM^i(t, x)\end{aligned}$$

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Representation theorems : [\[El Karoui & Méléard \(1990\)\]](#)

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Thank you for your attention !

Questions ?