Mean field limits for Hawkes processes in a diffusive regime

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Point process $=$ Jump process

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Hawkes processes = Interacting point processes on $\mathbb{R}_+$

Example : 2 processes Z_1 and Z_2

Z_1 inhibits Z_2
Z_2 self-excitation

Modeling in neurosciences

Neural activity $=$ Set of spike times

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	=	Point process

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Spike rate depends on the potential of the neuron

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Spike rate depends on the potential of the neuron

Each spike modifies the potential of the neurons

Contents

Stochastic Intensity

Z point process on $\mathbb{R}_+$

$\lambda \ : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ stochastic process

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Z point process on $\mathbb{R}_+$

$\lambda \ : \mathbb{R}_+ \to \mathbb{R}_+$ stochastic process

λ stochastic intensity of Z if :

$$\forall 0 \leq a < b, \mathbb{E}\left[Z([a,b])|\mathcal{F}_a\right] = \mathbb{E}\left[\int_a^b \lambda(t)dt \middle| \mathcal{F}_a\right]$$

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$(Z^1, \ldots, Z^N)$ system of Hawkes processes :

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$Z^i([0,t])$ = number of spikes of neuron i in $[0,t]$

$X_t^{N,i}$ = potential of neuron i at time t

f_i = spike rate function

h_{ji} = leakage function

Hawkes processes in diffusive mean field

For each $N \in \mathbb{N}^*$, we consider $\left(Z^{N,1}, \ldots, Z^{N,N}\right)$:

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$U_j(s)$ iid with mean 0 and variance 1

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Dynamique de X^N

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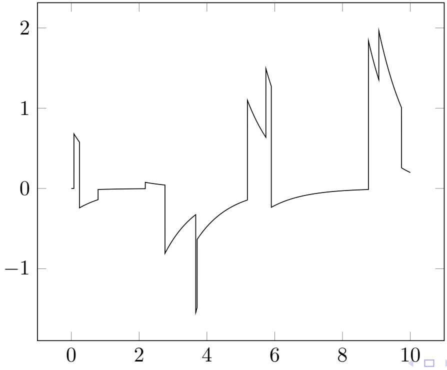
$$\begin{cases} X_t^N = X_s^N e^{-\alpha(t-s)} & \text{if none of the } Z^{N,j} \text{ charge } [s,t] \\ X_t^N = X_{t-}^N + \frac{U_j(t)}{\sqrt{N}} & \text{if } Z^{N,j} \text{ charges } t \end{cases}$$



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A generator of X :

$$Ag(x) \;:=\; \frac{d}{dt}\left(P_t g(x)\right)\Big|_{t=0}$$

Convergence of the generators

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$$d\bar{X}_t = -\alpha \bar{X}_t dt + \sqrt{f(\bar{X}_t)} dB_t$$

Convergence of the semigroups (1)

$$\left(\bar{P}_t - P_t^N\right) g(x) = \int_0^t P_{t-s}^N \left(\bar{A} - A^N\right) \bar{P}_s g(x) ds$$

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$$\begin{aligned}\left(\bar{P}_t - P_t^N\right) g(x) =& u(t) - u(0) \\ =& \int_0^t u'(s) ds \\ =& \int_0^t \left[-\frac{d}{du} \left(P_u^N \bar{P}_s g(x) \right) \Big|_{u=t-s} + \frac{d}{du} \left(P_{t-s}^N \bar{P}_u g(x) \right) \Big|_{u=s} \right] ds\end{aligned}$$

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=& \int_0^t u'(s) ds \\
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=& \int_0^t \left[-P_{t-s}^N A^N \bar{P}_s g(x) + P_{t-s}^N \bar{A} \bar{P}_s g(x) \right] ds
\end{aligned}$$

Convergence of the semigroups (2)

$$\left(\bar{P}_t - P_t^N\right) g(x) = \int_0^t P_{t-s}^N \left(\bar{A} - A^N\right) \bar{P}_s g(x) ds$$

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&\leq \int_0^t \mathbb{E}_x \left[\left|\left(\bar{A} - A^N\right) \bar{P}_s g(X_{t-s}^N)\right|\right] ds \\
&\longrightarrow 0
\end{aligned}$$

Convergence in finite-dimensional distribution

Convergence of the semigroups :

$$\mathbb{E}_x\left[g\left(X_t^N\right)\right] \longrightarrow \mathbb{E}_x\left[g\left(\bar{X}_t\right)\right]$$

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Induction + classical argument of Markov theory

$\Longrightarrow$ Convergence in finite-dimensional distribution :

$$\mathbb{E}_x\left[g_1\left(X_{t_1}^N\right)\dots g_n\left(X_{t_n}^N\right)\right] \longrightarrow \mathbb{E}_x\left[g_1\left(\bar{X}_{t_1}\right)\dots g_n\left(\bar{X}_{t_n}\right)\right]$$

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- X^N converges in fidi distribution to $\bar{X}$
- $\{X^N \ : \ N \in \mathbb{N}^*\}$ tight on $D(\mathbb{R}_+, \mathbb{R})$ (admited)

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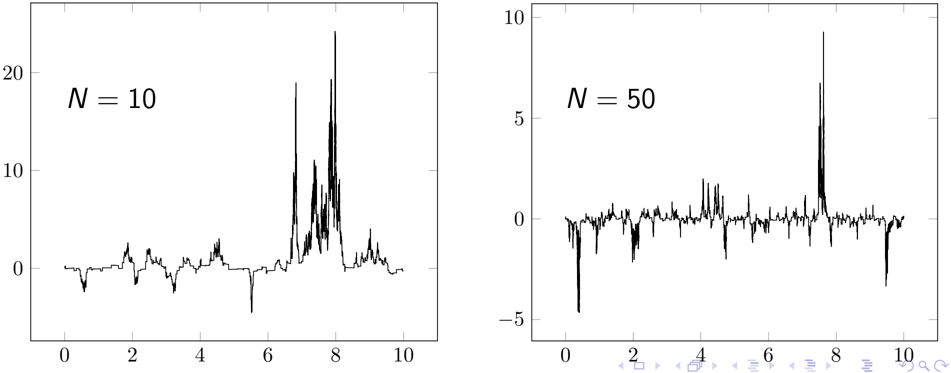

Convergence of $Z^{N,i}$

$$Z_t^{N,i} := \int_{]0,t]\times\mathbb{R}_+} 1_{\{z \leq f(X_{s-}^N)\}} d\pi_i^N(s,z)$$

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$$\bar{Z}_t^i := \int_{]0,t]\times\mathbb{R}_+} 1_{\left\{z\leq f(\bar{X}_{s-})\right\}} d\bar{\pi}_i(s,z)$$

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$$Z_t^{N,i} := \int_{]0,t]\times\mathbb{R}_+} 1_{\{z\leq f(X_{s-}^N)\}} d\pi_i^N(s,z) = \Phi(X^N, \pi_i^N)(t)$$

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Skorohod's Representation Theorem :

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$$\begin{array}{ccc} (X^N,\pi_i^N) & \xrightarrow{\mathcal{L}} & (\bar{X},\bar{\pi}_i) \\ \mathcal{L}\ \Vert & & \Vert\ \mathcal{L} \\ (\widetilde{X}^N,\widetilde{\pi}^N) & \xrightarrow{as} & (\widetilde{X},\widetilde{\pi}) \end{array} \implies \Phi(\widetilde{X}^N,\widetilde{\pi}^N) \xrightarrow{as} \Phi(\widetilde{X},\widetilde{\pi})$$

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$$\begin{array}{ccccccc} (X^N, \pi_i^N) & \xrightarrow{\mathcal{L}} & (\bar{X}, \bar{\pi}_i) & & \Phi(X^N, \pi_i^N) & \xrightarrow{\mathcal{L}} & \Phi(\bar{X}, \bar{\pi}_i) \\ \mathcal{L} \parallel & & \parallel \mathcal{L} & \Longrightarrow & \mathcal{L} \parallel & & \parallel \mathcal{L} \\ (\widetilde{X}^N, \widetilde{\pi}^N) & \xrightarrow{as} & (\widetilde{X}, \widetilde{\pi}) & & \Phi(\widetilde{X}^N, \widetilde{\pi}^N) & \xrightarrow{as} & \Phi(\widetilde{X}, \widetilde{\pi}) \end{array}$$

Result : $(Z^{N,i})_{i\ge1}$ converges to $(\bar{Z}^i)_{i\ge1}$ in distribution in $D(\mathbb{R}_+, \mathbb{R})^{\mathbb{N}^*}$

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Thank you for your attention !

Questions ?